Scaling Relations in Elastic Turbulence

a review on Victor Steinberg’s paper
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Observation of elastic turbulence – setup and stress increase

• Flow of a sufficiently elastic polymer solution can become irregular even at low velocity, high viscosity and in a small tank. An increase in the flow resistance by a factor of about twenty is observed, while the Reynolds number is arbitrarily low. This flow has all the main features of developed turbulence.

• Curves 1 and 2 for polymer solution with gap width of \( d = 10 \text{ mm} \) and \( d = 20 \text{ mm} \). Curve 3 belongs to pure solvent.

• The ratio \( \sigma / \sigma_{\text{lam}} \) keeps growing with the shear rate, and at the highest \( \dot{\gamma} \) that has been reached the flow resistance is about 12 times larger than is found in laminar flow. In the same range of shear rates, flow of the pure solvent is completely laminar and the ratio \( \sigma / \sigma_{\text{lam}} \) is unity within the resolution of the viscometer (about 1%).

\[ \dot{\gamma} = \omega R / d \]
Power spectral density shows how power of a signal is distributed over a range of frequencies.

\[
P = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} |x(t)|^2 dt
\]

Power spectral density can be defined for spatial or temporal velocity, pressure, concentration, energy, etc. fluctuations: For a signal \(x(t)\) with finite length \([0,T]\), the truncated Fourier transform can be defined as,

\[
\hat{x}(\omega) = \frac{1}{\sqrt{T}} \int_{0}^{T} x(t)e^{-i\omega t} dt
\]

Then, according to definition of signal power, we can define PSD,

\[
PSD(\omega) = \lim_{T \to \infty} \mathbb{E} \left[ |\hat{x}(\omega)|^2 \right]
\]

where \(\mathbb{E}\) is the expected value.
Observation of elastic turbulence – spectral decay

- Flow velocity was measured in the horizontal plane in the centre of the set-up, where its average value is zero. As the shear rate is raised the power of fluctuations increases and characteristic frequencies become higher, but the general form of the spectra remains very much the same. Numbers 1 to 5 refer to shear rates increasing from 1.25 to 5.9 [1/s].

- Polymer molecules are stretched in the primary shear flow, which makes it unstable and causes irregular secondary flow. This flow acts back on the polymer molecules, stretching them further and becoming increasingly turbulent, until a kind of saturated dynamic state is reached. The density of the elastic energy of the stretched polymers increases by about the same factor as the elastic stress, while the kinetic energy remains small.
Defining power spectra based on the definition above for different parameters:

Kinetic energy spectra: \( E(f) = \left| \langle \mathbf{A}(f) \rangle \right|^2 \)

We saw in Steinberg 2000 paper that \( \langle |\mathbf{A}(f)|^2 \rangle \sim f^{-\alpha} \)
(power law decay or line of slope \(-\alpha\) in log-log plot)
and we know experimentally that \( \alpha \approx 3.5 \).

Pressure power spectra: \( E_{pp} = \langle |\mathbf{P}(f)|^2 \rangle \sim f^{-\beta} \)

Having the 2000 experiments in mind, torque is directly related to averaged stress.
If \( \langle \tau_w(r,t) \rangle \) is the average stress on driving disk,
\[
\Gamma(t) = \int_0^R \langle \tau_w(r,t) \rangle 2\pi r^2 dr
\]
\( \langle \tau_w(r,t) \rangle \) can be written as sum of averaged elastic stress \( \langle \tau_{ij} \rangle \) and averaged viscous stress \( \langle \sigma_{\text{visc}} \rangle \)
\[
\langle \tau_w(r,t) \rangle = \langle \tau_{ij} \rangle + \langle \sigma_{\text{visc}} \rangle
\]
If \( \text{Re} \ll 1 \) and \( Wi \) is low (below onset of elastic instability) \( \langle \tau_w(r,t) \rangle = \tau_w = \langle \sigma_{\text{visc}} \rangle \)
(Independence of \( r \) and \( t \) )
\[ P = \frac{2\pi R^3}{3} w + f(r,t) \]

If \( \text{Re} \ll 1 \) and \( \text{Wi} \ll 1 \) (remember \( \sigma / \sigma_\text{tan} \) plot in 2000 paper) \( \sigma_{ij} \gg \sigma_{\text{visc}} \)

\[ \langle \tau_w (r,t) \rangle = \langle \sigma_{ij} \rangle + \frac{\langle \sigma_{\text{visc}} \rangle}{\infty} \approx \langle \sigma_{ij} \rangle \]

Assuming \( \sigma_{ij} \) is independent of \( r \) (approximation)

\[ \langle \sigma_{ij} \rangle = \langle \sigma_{ij} (r) \rangle \rightarrow \Gamma (r) \sim \langle \sigma_{ij} (r) \rangle \]

**Torque power spectra:**

\[ E_{\Gamma \Gamma} = \langle | \hat{\Gamma}(f) |^2 \rangle \]

\[ \sim \langle | \hat{\sigma}(f) |^2 \rangle \]

\[ \sim f^{-\mu} \]

* Key goal of this paper is to find scaling relations between \( \alpha, \beta, \) and \( \mu \) — decay rate of torque spectra, decay rate of pressure spectra, decay rate of energy spectra.

To achieve this goal, let's consider the governing equations:

**Continuity for incompressible flow,**

\[ \nabla_i V_i = 0 \]

**Causky momentum equation without inertia terms** (\( \text{Re} \ll 1 \))
\[ \nabla \cdot \rho = \nabla \cdot \Delta V_i + F_i; \quad \Delta = \nabla^2 \quad \rho \rightarrow \text{dynamic viscosity} \]

- pressure gradient term
- elastic stresses term
- viscous stresses term

Elastic stresses term: \( F_i = \nabla_j \sigma_{ij} \)

Oldroyd-B constitutive equation for elastic stresses:
\[ \frac{d\sigma_{ij}}{dt} = \sigma_{jkl} \nabla_k V_i + \sigma_{ik} \nabla_k V_j - 2\sigma_{ij} / \lambda \]

in 2.341 notation:
\[ \Xi + \lambda \Xi(1) = -Z \left[ \Xi(1) + \frac{\lambda^2 s}{2s + \lambda \rho} \Xi(2) \right] \]

With a bit of algebra it can be seen that they are the same. In class we did not separate viscous stress from elastic stress and wrote the equation based on \( \Xi(1) \) and \( \Xi(2) \).

\* ET is observed when polymer coils are sufficiently stretched, i.e., \( \sigma_{ij} \gg \sigma_{ij} \) \( \rightarrow \) elastic stress due to thermal noise.

This means ET exists at \( R_i \gg R_o \)
end to end vector \( \rightarrow \)
for a given configuration of a polymer chain
\( R_i \rightarrow \)
polymer random coil \( \sim R_g \)
radius of gyration \( R_o \).
Thus, \( \sigma_{ij} \sim \langle R_i R_j \rangle = R_i R_j \) [m^2] \( \sigma_{ij} = NKT \langle R_i R_j \rangle \)

\* Here comes the interesting analogy between polymer hydrodynamics and magneto hydrodynamics

\( \sigma_{ij} \sim \langle R_i R_j \rangle \sim R_i R_j \)

\( \sigma_{ij} = B_i B_j \) (at zero magnetic resistivity)

equivalent of flexible polymers

\( \Rightarrow \) Cauchy momentum equation turns to

\[ \nabla_i \rho = \nabla_j (B_i B_j) \]

\[ \nabla_j (\sigma_{ij}) \]

\( B \) has solenoidal nature \( \Rightarrow \nabla_i B_i = 0 \)
equivalent of maximum length of polymer chains is constant.

\* We define torque power spectra as

\[ E_{pp} \sim \langle (\tilde{\omega})^2 \rangle \sim f^{-\alpha} \Rightarrow \langle (\tilde{\omega})^2 \rangle \sim f^{-\alpha/2} \]

\[ \sim f^{-\gamma} \]

\* Now let's find the first scaling relation between \( \alpha \) and \( \gamma \)

\[ \Rightarrow \langle (\tilde{\omega})^2 \rangle \sim f^{-\gamma} \]

\[ \Rightarrow \langle (\tilde{\omega})^2 \rangle \sim f^{-\alpha} \text{ (exp. found to be } -3.5) \]
\[ \sigma_{ij} = B_i B_j \sim R_i R_j \]

For simplicity, consider plane Couette flow with uniform shear rate where polymer is introduced.

\[ R_i \sim R_o \nabla V \]

\[ \Rightarrow \quad E_B = E_o = \left< | R_i R_j |^2 \right> \sim \left< \nabla V \cdot \nabla V \right> \left[ \text{energy} \left/ \text{kg} \cdot \text{s}^2 \right. \right] \]

\[ = f^{-(\alpha-2)} \]

\[ E(f) = \left< V V \right> f \sim f^{-\alpha} \left[ \text{energy} \left/ \text{kg} \right. \right] \]

\[ \Rightarrow \quad V = \alpha - 2 \quad \text{and} \quad \mu = 2(\alpha - 2) \]

*Now, we need to find relation between \( \alpha \) and \( \beta \)*

To this end we use the Cauchy momentum equation we wrote earlier with our analogy to magnetohydrodynamics.

\[ \nabla_i \rho = 2 \nabla^2 V_i + \nabla_j (B_i B_j) \quad \text{eq. 1} \]

\[ \Rightarrow \quad -\nabla_j (B_i B_j) = -\nabla_i \rho + 2 \nabla^2 V_i \]

Taking the Fourier transform from both sides,
\[-i \kappa \beta^2 \kappa \beta^{-2} = -i \kappa \rho \beta^{-1/2} \frac{\kappa}{\beta^{-2}} + i^2 \gamma (\kappa^2 \beta^{-2})^{-1/2} \]

Simplifying:
\[i \beta^2 (\kappa^2 \beta^{-2}) = i \beta (\kappa^2 \beta^{-1})^{-1/2} \]

\[+ \gamma (\kappa^2 \beta^{-2})^{-1/2} \]

\[\Rightarrow i \beta^2 (\kappa^2 \beta^{-2}) = i \beta (\kappa^2 \beta^{-1})^{-1/2} \gamma (\kappa^2 \beta^{-2})^{-1/2} \]

Note that here \( \kappa \) is the wave number and \( L \) is the length scale that defines the largest average velocity gradient.

* Now, taking divergence of equation 1:
\[\nabla \cdot (\nabla \rho) = \nabla \cdot (\gamma (\kappa^2 \beta^{-1}) + \nabla (\kappa^2 \beta^{-1})) \]

\[\Rightarrow \nabla^2 \rho = \gamma (\kappa^2 \beta^{-1}) \frac{\partial \beta \beta^{-1}}{\partial x_i \partial x_j} \]

Taking Fourier transform:
\[E_\beta (\kappa L) \gamma (\kappa^2 L^2 \beta^{-2}) \sim E_\beta (\kappa L) \gamma (\kappa^2 L^2 \beta^{-2}) \]

\[\Rightarrow E_\beta (\kappa L) \sim E_\beta (\kappa L) \sim \beta (\kappa L)^{-1/2} \sim \beta (\kappa L)^{-2} \]

\[\Rightarrow \beta = 2 (\kappa - 2) \]
Scaling Relations in Elastic Turbulence

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We report the scaling relations between the exponents of the power-law decays of kinetic and elastic energy spectra, pressure, as well as torque fluctuations in elastic turbulence (ET). The relations are derived by estimating that the divergent part of the elastic stress is much larger than its vortical part, and its contribution into the full elastic stress is dominant in the range of the power spectrum amplitudes observed experimentally in ET. The estimate is in line with polymer stretching by flow: the polymers are stretched mostly by the divergent part associated with a strain rate, whereas a rotational, or vortical, flow plays a minor role in the polymer stretching. The scaling relations agree well with the exponent values obtained experimentally and numerically in the ET regime of a viscoelastic fluid in different flow geometries.

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Elastic turbulence (ET) is a chaotic flow at low Reynolds number, Re \( \ll 1 \), and large Weissenberg number, Wi = \( \lambda V/l \gg 1 \), discovered two decades ago and currently well studied and characterized experimentally, theoretically, and numerically in various flow configurations with curvilinear streamlines. Here \( \lambda \) is the longest polymer relaxation time, \( V \) is the average flow velocity, and \( l \) is the characteristic length scale. ET displays large velocity fluctuations in a wide range of spatial and temporal scales, a significant growth of flow resistance, and many orders of magnitude enhancement of mixing efficiency [1–3]. Elastic instabilities and ET at Re \( \ll 1 \) are driven solely by elastic stresses generated by polymer stretching caused by the flow. The key observation in ET is the power-law decay of the kinetic energy spectra in a frequency domain \( E(f) = \langle \hat{u}^2(f) \rangle \sim f^{–\alpha} \) with the exponent \( \alpha > 3 \) (between 3.3 and 3.6 depending on flow geometry) [1–4]. Since the validity of the Taylor hypothesis is justified with some limitations also in ET [5], one expects the same exponent value in both the f and k domains, that has been indeed confirmed via particle image velocimetry (PIV) in a swirling flow between rotating and stationary disks where \( \alpha \approx 3.5 \) in the k domain is measured [6].

Further on, additional key features of ET in a swirling flow are identified such as the power-law decays of the power spectra of torque \( \Gamma \), \( E_{\Gamma} = \langle \hat{\Gamma}^2(f) \rangle \), and the pressure \( p \), \( E_{pp} = \langle \hat{p}^2(f) \rangle \). Both power spectra are characterized by a flat region at \( f/f_{vor} < 1 \) and the steep power-law decay at \( f/f_{vor} > 1 \) scaled as \( E_{\Gamma} \sim (f/f_{vor})^{–\mu} \) [7,8] and \( E_{pp} \sim (f/f_{vor})^{–\beta} \) [7,8] at polymer concentration \( \phi = 100 \text{ ppm} \) and various Wi and for various \( \phi \ll 900 \text{ ppm} \) at the maximum Wi values with the exponents \( \mu \approx 4 \) and \( \beta \approx 3 \), respectively. Here \( f_{vor} \) is the main vortex frequency with a distinctive peak at \( f/f_{vor} = 1 \) in the spectra [3,7,8]. One should emphasize a surprising repeatability and reliability of the value of \( \beta \approx 3 \) obtained in ET in different flow geometries: a swirling flow between two disks, a wake between two widely separated obstacles hindering a channel flow, and a flow past a cylinder [4,7–10]. Moreover, in the swirling flow at \( \phi = 300 \text{ ppm} \) and Wi = 100 with more than \( 10^5 \) data points taken in the separate experiment, one gets a very reliable result of \( \beta = 3 \pm 0.1 \) [see [8] Fig. 14(b)]. It is distinctly different from experimental values of \( \alpha \) varied in the range 3.3–3.6 depending on flow geometry and in the range 3.2–3.8 in the numerical simulations.

One can show that in ET of a swirling flow, the torque is directly related to the averaged elastic stress, \( \langle \sigma_{ij} \rangle \), on the driving disk. In general, \( \Gamma(t) = \int_0^\infty \langle \tau_w(r,t) \rangle 2\pi r^2 dr \), where \( \langle \tau_w(r,t) \rangle \) is the average stress on the driving disk. At Re \( \ll 1 \) and Wi below an elastic instability onset, \( \langle \tau_w(r,t) \rangle = \tau_w = \langle \sigma_{visc} \rangle \), where \( \langle \sigma_{visc} \rangle \) is the averaged viscous shear stress on the disk that is independent of \( r \) and related to the torque as \( \Gamma = (2\pi R^3/3)\tau_w \). In the ET regime at Re \( \ll 1 \) and Wi \( \gg 1 \), \( \langle \tau_w(r,t) \rangle = \langle \sigma_{ij} \rangle + \langle \sigma_{visc} \rangle \). Moreover, in ET \( \sigma_{visc} \ll \sigma_{ij} \) and so \( \langle \sigma_{visc} \rangle \ll \langle \sigma_{ij} \rangle \) [11,12]. Then one gets \( \langle \tau_w(r,t) \rangle \approx \langle \sigma_{ij}(r,t) \rangle \). Thus in the approximation of the independence of \( \langle \sigma_{ij}(r,t) \rangle \) on \( r \), one obtains \( \Gamma(t) \sim \langle \sigma_{ij}(t) \rangle \) that leads to \( E_{\Gamma} = \langle \hat{\Gamma}^2(f) \rangle \sim \langle \hat{\sigma}(f) \rangle^2 \sim f^{–\mu} \).

To explain the power-law behavior and the scaling exponents of the main flow parameters in ET, theory [13] and numerical simulations [14,15] are carried out for an isotropic, homogeneous, and unbounded flow of a dilute polymer solution described by the Oldroyd-B model with linear polymer elasticity [16] and with a back reaction of the stretched polymers on the flow. Further on, the numerical simulations in a wall-bounded channel flow with...
a linear periodic array of obstacles [17], and in two-dimensional Taylor-Couette flow geometry [18] of a dilute polymer solution characterized by the same Oldroyd-B model at low Re are employed to test the power-law decay exponent of the velocity power spectrum in ET. In all these simulations similar results of \( \alpha > 3 \) with variations between about 3.2 and 3.8 are found in accord with the experiment and theory. Another 2D direct numerical simulation (DNS) of forced fluid film flow of a polymer solution described by a finite extension nonlinear elasticity-Peterlin (FENE-P) polymer model [16] at Re \( \ll 1 \) and Wi \( \gg 1 \) reveals a chaotic flow characterized by the decay of the kinetic energy spectrum in a \( k \) domain with \( \alpha \approx 3.2 \) [19], in a good agreement with the experiment [1,2], theory [13], and other numerical simulations [14,15,17,18] in ET. One of the key conclusions of DNS is the independence of the exponent value of the power-law decay of the kinetic energy spectrum on the polymer model. The theory of ET is based on the Navier-Stokes equation (NSE) for a polymer solution with a coupling term, engendered by the back reaction of the elastic stress field \( \sigma_{ij} \) on the velocity field \( V_i \), and the constitutive equation for \( \sigma_{ij} \) in the Oldroyd-B model approximation [16]. In ET at Re \( \ll 1 \), an inertial term in NSE can be neglected. For sufficiently stretched polymers, one gets \( \sigma_{ij} \gg \sigma^0_{ij} \), where \( \sigma^0_{ij} \) is the elastic stress due to thermal noise, and two equations describing polymer hydrodynamics in the case of the Oldroyd-B model with the incompressibility condition are [16]

\[
\nabla_i p = \eta \Delta V_i + F_i, \tag{1}
\]

\[
F_i = \nabla_j \sigma_{ij}, \tag{2}
\]

\[
\nabla_i V_i = 0, \tag{3}
\]

\[
d_{ij} \sigma_{ij} = \sigma_{jk} \nabla_k V_i + \sigma_{ik} \nabla_k V_j - 2 \sigma_{ij}/\lambda, \tag{4}
\]

where \( \eta \) is the dynamic viscosity. It was proved in [13] that in a statistically steady state realized at times much longer than the Lagrangian velocity gradient correlation time the polymer stress tensor \( \sigma_{ij} \) is relaxed to the uniaxial form \( \sigma_{ij} = B_i \delta_{ij} \), if \( \sigma_{ij} \gg \sigma^0_{ij} \), so \( R_i \gg R_0 \) is considered, where \( R_i \) is the end-to-end vector for a given configuration of a polymer \( i \), and \( R_0 \) is a radius of polymer random coil at equilibrium (gyration radius [16]). Then \( \sigma_{ij} \sim \langle R_i R_j \rangle \sim R_i R_j \), as shown in [13]. The uniaxial form of \( \sigma_{ij} \) allows us to rewrite the set of equations for polymer hydrodynamics in the form similar to magnetohydrodynamic (MHD) equations [20] with \( \sigma_{ij} = B_i B_j \) at zero magnetic resistivity [13], from which only the Stokes equation is presented below

\[
\nabla_i p = \eta \Delta V_i + \nabla_j (B_i B_j), \tag{5}
\]

where \( B_i \) has the solenoidal nature, characterizes the degree of the coherent polymer extension, and, in contrast to the magnetic field vector in MHD, is defined up to a sign, analogously to the director in nematic liquid crystals [13]. Another important difference with MHD appears in the relaxation term instead of the magnetic resistivity [20]. At the sufficiently large back reaction of the elastic stress determined by the term \( (B_i \nabla_j) B_j \), the flow reaches a chaotic, statistically steady state defined as ET.

The major result of the theory based on Eqs. (1–4) is the prediction of the power-law decay of the spherically normalized kinetic energy spectrum \( E(k) \sim V^2 \rho l^2(k)^{-\alpha} \) with \( \alpha > 3 \) and the related to it the elastic energy spectrum \( E_B(k) \sim B^2 l(k)^{-\alpha} \), where \( \nu = (\alpha - 2) > 1 \) that is close to the passive scalar Batchelor decay exponent \( -1 \) [21]. Here \( k \) is the wave number, and \( l \) is the length scale that defines the largest average velocity gradient. The exponents of both power spectrum decays are obtained by applying a passive scalar approach for a description of the small scale perturbations of the velocity \( \vec{u} \) and elastic stress \( \vec{b} \) fields passively advected by the large scale random velocity \( V_i \) field, where \( u_i, b_i \ll V_i, B_i \) and \( \nabla_i u_i \ll \nabla_i V_i \), while \( \nabla_i b_j \sim \nabla_i B_j \) [13]. From both of the power spectrum expressions, one finds that the main energy in ET is carried out by the stretched polymers resulted in the elastic energy \( \sim B^2 \). The latter follows from the relation \( E(k) \sim Re(k)^{-2} E_B(k) \), where \( \rho (V/B)^2 \sim Re \ll 1 \), since \( B^2 \sim \eta/\lambda \) and \( \nu V^2 \sim \rho l^2/\lambda \eta \) that leads to \( \rho (V/B)^2 \sim \rho l^2/\lambda \eta \). The theoretical value of \( \alpha \) is found in good accord with the experimental results [1–4,6].

The same relation in ET between \( \nu \) and \( \alpha \) can also be obtained just by scaling arguments considering \( \sigma_{ij} = B_i B_j \sim R_i R_j \) leading to \( B_i \sim R_i \). A statistically steady state of a polymer extension is determined by the interplay between a stretching due the velocity difference between the polymer ends, estimated by the velocity gradient multiplied on \( R_i \), and its relaxation. To simplify the issue, we consider a plane Couette flow with a uniform shear rate where a polymer is introduced. The polymer is stretched in the shear direction up to \( R_i \) and remains unchanged in the perpendicular one \( R_i \approx R_0 \) over the relaxation time \( \lambda \) that leads to a relation \( R_i = R_i \nabla \vec{V} \lambda \) (see [22]). Thus taking into account that \( R_i \sim \nabla \vec{V} \), one finally obtains \( E_B(f) \sim \langle \nabla \nabla \nabla \vec{V} \rangle_f \sim f^{-(\alpha - 2)} \), since \( E(f) \equiv \langle \vec{V} \nabla \vec{V} \rangle_f \sim f^{-\alpha} \), resulting in the equation \( \nu = \alpha - 2 \), equivalent to the relation
obtained by analytical calculations in [13] and discussed above.

Though the theory of ET predicts the first relation between the scaling exponents of the elastic and kinetic energy spectrum decays \( \nu = \alpha - 2 \) [13], it does not provide any clue on the relation between the scaling exponents of the kinetic energy spectrum and pressure power spectrum decays \( \alpha \) and \( \beta \), respectively.

Using the Stokes equation [see Eq. (5)], we derive in ET the relation between the Fourier transforms of the pressure \( E_p(k) \sim k^{-\beta/2} \), velocity \( E_u(k) \sim k^{-\alpha/2} \), and the elastic stress \( E_B(k) \sim k^{-\nu} \), where \( \nu = \alpha - 2 \) is used,

\[ iB^2(k)(k)^{-(\alpha-2)} = i p(k)(k)^{-\beta/2} + \eta(k)VL^{-1}(k)(k)^{-(\alpha-2)/2}. \]  

(8)

The Eq. (8) presents the elastic stress power spectrum as a sum of two terms. By straightforward calculations we demonstrate that the first term on the right side of Eq. (8) is the power spectrum of the divergent part of the elastic stress \( \sigma_{ij}^{\text{div}} \equiv \langle B_i B_j \rangle^{\text{div}} \). Indeed, by taking a divergence operation on the both sides of Eq. (5), one gets \( \Delta p = \partial^2(B_i B_j)^{\text{div}}/\partial x_i \partial x_j \) [7,8], which Fourier transform leads to the relation \( E_B(k)^{\text{div}} \sim E_p(k) \sim p(k)^{\beta/2} \) after canceling \( (k)^2 \) in both terms. Then one finds that \( E_B(k)^{\text{div}} \sim p(k)^{\beta/2} \) is equivalent to the first term on the right side of the Eq. (8) after canceling factor \( k \) in all three terms. Similarly, by taking a curl operation of the same Eq. (5) provides the exact relation between the vortical (divergence-free) part of the elastic stress \( \sigma_{ij}^{\text{curl}} \equiv \langle B_i B_j \rangle^{\text{curl}} \) and the vorticity \( \vec{\Omega} = \vec{\nabla} \times \vec{v} : \vec{\nabla} \times \vec{\nabla}(\vec{B} \cdot \vec{B})^{\text{curl}} = -\eta \Delta \vec{\Omega} \) [8]. Then the Fourier transform results in the relation \( E_B(k)^{\text{curl}} \sim \eta V L^{-1}(k)^{-(\alpha-2)/2} \) that is equivalent to the second term on the right side of Eq. (8) after canceling the factor \( k \). Thus, the Eq. (8) can be rewritten as \( E_B(k) = E_B(k)^{\text{div}} + E_B(k)^{\text{curl}} \), and after substituting the expressions for \( E_B(k)^{\text{div}} \) and \( E_B(k)^{\text{curl}} \) and performing straightforward algebraic transformations one gets the following algebraic equation

\[ B^2(k)^{\text{2}} = p x^{\beta/(\alpha-2)} - i(\eta V L/x), \]  

(9)

where \( x = (k)^{-(\alpha-2)/2} \). In ET at \( kl < 1 \) or \( \lambda k < 1 \) (and so \( x > 1 \)) just before the power-law decay onset, the elastic energy spectrum is flat and the amplitude ratio of the Fourier transforms \( E_B(k)^{\text{div}}/E_B(k)^{\text{curl}} \) is large due to both \( x > 1 \) and the large ratio of the coefficients of two terms on the right side of Eq. (9), namely \( pl/\eta V \gg 1 \). Regarding the second inequality, by neglecting the last term due it smallness one gets the equality \( p = B^2 \). Then by using \( B^2/\eta V \), one can rewrite it as \( (B^2/\rho V^2)(\rho V/\eta) = El(\rho V/\eta) = El \cdot Re = Wi \gg 1 \). Then the ratio \( E_B(k)^{\text{div}}/E_B(k)^{\text{curl}} \gg 1 \) at \( kl < 1 \) and even at \( kl > 1 \) up to at least \( kl \approx 10 \) in spite of a decay of \( E_B(k)^{\text{div}} \) steeper than of \( E_B(k)^{\text{curl}} \) at \( kl > 1 \) (or \( \lambda k > 1 \)). Thus, the last term on the right side of Eq. (9) can be neglected in the whole range of the power-law decay of the energy spectrum obtained in the experiment. As the result, one gets a new scaling relation:

\[ \beta = 2(\alpha - 2). \]  

(10)

Thus, the conclusion is that \( \sigma_{ij}^{\text{div}} \) plays the dominant role in the elastic stress field, whereas \( \sigma_{ij}^{\text{curl}} \) has minor contribution into polymer stretching, and so the vortical part of the elastic stress in ET can be neglected in the range of the velocity and pressure power spectra found experimentally [14,6,10]. Moreover, this conclusion is in line with the physics of the polymer stretching in ET: polymers are considerably stretched by the flow strain rate reflected by the divergent part of the elastic stress, whereas in rotational (vortical) flow polymers remain almost unstretched. This conclusion is recently confirmed and clarified both theoretically and numerically by considering a novel effect of preferential sampling of elastic chains in turbulent flows. It is demonstrated numerically that, first, the stretched chains are located in the flow regions with low vorticity, and, second, the elastic chains are trapped in the vortical regions, where they are found in a coil state with the negligible contribution into the elastic energy [23].

Using the new scaling relation [Eq. (10)] and the experimental value of \( \alpha \approx 3.5 \), one gets for the power-law decay exponent of the pressure power spectrum \( \beta \approx 3 \) in good agreement with the experiment [4,7–10], and for the algebraic decay exponent of the elastic stress power spectrum \( \nu \approx 1.5 \), which is still not verified in ET either experimentally or numerically. On the other hand, from the above suggested relation between the torque and averaged over the upper driving disk elastic stress power spectra \( E_{\text{IT}} = \langle \hat{f}(f)^2 \rangle / \langle \sigma(f)^2 \rangle \sim f^{-\mu} \) one gets the relation \( \mu = 2\nu \). In this case, taking the obtained above value \( \nu \approx 1.5 \), one finds \( \mu \approx 3 \) instead of the experimental value \( \mu \approx 4 \) [7,8], where the discrepancy could be attributed to the assumptions, first that \( \langle \sigma_{ij}(r,t) \rangle \) is independent of \( r \) to get the relation \( \Gamma(t) \sim \langle \sigma_{ij}(t) \rangle \) and second that the scaling exponent for \( \sigma_{ij} \) is the same used in the derivation of \( \mu = 2\nu \).

Another well-known example of a chaotic, spatially smooth and random in time flow is a high-Re inertial turbulence in a deep-dissipation scale range, in Newtonian fluid turbulence. In this case at scales \( k^{-1} < \zeta \), the kinetic energy spectrum decays exponentially, and the velocity field is characterized by the single spatial scale \( \zeta \) [24]. There is significant evidence coming from 3D DNS of homogeneous and isotropic viscoelastic turbulence in a turbulent drag reduction (TDR) regime that the fluid elasticity qualitatively modifies the velocity power spectrum at scales \( k^{-1} < \zeta \) due to a strong increase of a kinetic energy content caused by polymer stretching [22,25–30].
It has been demonstrated by 3D DNS of a simplified viscoelastic fluid model that in the inertial range, at large scales, a turbulent energy cascade leading to the kinetic energy power spectrum remains roughly unchanged by the presence of polymers, whereas at small scales, below \( \zeta \), the velocity power spectrum decay is modified from exponential for Newtonian fluid to algebraic for viscoelastic fluid, though the flow remains chaotic in time and spatially smooth\[22,25–30\].

The first quantitative results on the power-law decay \( k^{-\alpha} \) of the kinetic energy power spectrum with a detailed analysis of the dependence of \( \alpha \) on Wi and Re are obtained by numerical calculations using a hybrid Eulerian-Lagrangian approach to 3D homogeneous, isotropic, decaying turbulence in a polymer solution characterized by the FENE-P constitutive equation and at rather low initial Re (or at the turbulent Reynolds number \( R_e(t) \approx 52 \) based on the Taylor microscale) to attain larger \( \zeta \) value [see Figs. 5, 6, 27, Fig. 30(a) 28, and Figs. 5, 6 in 29]. At the reduced time \( t^* = 20 \) with \( R_e(t^* = 20) = 3.5, 3.2, 3.1, 3.0 \), and at four values of Wi = 25, 50, 100, 200, it is found that \( \alpha \) decreases from 4.6 down to 4.1 in the deep-dissipation range of scales, as Wi increases from 25 up to 200 [27,29]. However, the power spectrum amplitudes increase with Wi and significantly exceed that of Newtonian fluid [28]. Moreover, the same calculations reveal the pressure and elastic energy power spectrum decays with the exponents \( \beta = 3 \pm 0.2 \) and \( \nu = 2.2 \pm 0.1 \), respectively [29]. The authors of Ref. [29] have also undertaken an attempt to derive a scaling relation between \( \alpha \) and \( \beta \), though the suggested relation, which is different from that discussed above in this Letter, is not consistent with their numerical results.

Later on, a role of polymers in altering small scale dynamics of homogeneous, isotropic turbulence at Re \( \gg 1 \) is investigated by 3D DNS, and a short review of former relevant numerical results and a possible connection to ET are provided in Ref. [22]. The early results [25,26,30,31] on the small scale dynamics in TDR at Re \( \gg 1 \) in the presence of polymers are presented in a qualitative way only. So the early studies are conducted via 3D DNS in homogeneous, isotropic, stationary turbulence at moderate Re \( \sim 87 \) using the linear polymer Oldroyd-B model. They reveal that the kinetic energy spectrum is partially suppressed by energy transfer to polymers at \( k^{-1} < \zeta \), contrary to a significant increase found in other studies [26,30,32], and the kinetic energy spectrum demonstrates an algebraic decay with \( \alpha \approx 3.5 \) instead of an exponential decay for Newtonian fluid [25]. The value of \( \alpha \) is close to that found in other studies [25,26,30]. Another 3D DNS calculations carried out at low elasticity and moderate Re in a homogeneous, isotropic, decaying turbulence using the FENE-P polymer model report a significant decrease of the dissipation energy rate and an amplification of the kinetic energy and enstrophy spectrum amplitudes in the deep-dissipation range together with a slight simultaneous decrease of the kinetic energy spectrum in the inertial range [30]. Moreover, the kinetic energy spectrum shows power-law decay with \( \alpha \approx 3.5 \) at \( k^{-1} < \zeta \), close to that mentioned above [25]. Further progress in the DNS studies, where the scale-by-scale equation of the interaction between polymers and fluid flow is exploited, is presented in [31] and then later used in [33], where the scale-by-scale kinetic energy flux is analyzed. The latter reveals that the polymers remove energy from large scales and then transfer to the small scales where it dissipates. The energy balance between scales is further studied in [22], where the power-law decay exponent \( \nu \approx 2 \) of the elastic energy spectrum is clearly verified.

To summarize the results of the DNS investigations of homogeneous, isotropic turbulence at scales \( k^{-1} < \zeta \), one concludes that the values of the scaling exponents \( \alpha, \beta, \) and \( \nu \) are rather close to those found in ET and independent of the polymer model. However, a larger scatter in values for \( \alpha \) is found: between \( \alpha \approx 3.5 \) in DNS with various constitutive equations [25,26,30] in a good agreement with ET versus a hybrid Eulerian-Lagrangian DNS \( \alpha \) between 4.1 and 4.6 depending on Wi [27–29]. Such scatter in values of \( \alpha \) leads to the discrepancy between values of \( \beta \) and \( \nu \) obtained in DNS compared with those found from the scaling relations.

To conclude, the relations between the scaling exponents \( \alpha, \nu, \beta, \mu \) of \( \mathbf{V}, \mathbf{B}, p, \) and \( \Gamma \), are suggested and summarized in Table I. The relation between \( \alpha \) and \( \beta \) is well satisfied by the exponent values taken from either experiments or DNS in ET in various flow geometries, whereas the agreement between \( \alpha, \nu, \beta \) from DNS and the values obtained from the two scaling relations applied to high-Re viscoelastic turbulence at scales \( k^{-1} < \zeta \) is less satisfactory. This discrepancy between ET and high-Re viscoelastic turbulence may be caused by a different nature of the elastic stress field, namely the energy flux from the inertial to dissipation scales in latter versus its absence in ET. It can lead, for example, to a larger contribution of the vortical part of the elastic stress field \( E_B(kl)^{\text{curl}} \) in Eq. (9) than in ET, where \( E_B(kl)^{\text{curl}} \) is neglected in the derivation of the scaling relation in Eq. (10).

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( \nu )</th>
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<td>2(( \alpha - 2 ))</td>
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TABLE I. Table of the scaling exponent relations.
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Elastic turbulence in a polymer solution flow

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Turbulence is a ubiquitous phenomenon that is not fully understood. It is known that the flow of a simple, newtonian fluid is likely to be turbulent when the Reynolds number is large (typically when the velocity is high, the viscosity is low and the size of the tank is large\(^2\). In contrast, viscoelastic fluids\(^3\) such as solutions of flexible long-chain polymers have nonlinear mechanical properties and therefore may be expected to behave differently. Here we observe experimentally that the flow of a sufficiently elastic polymer solution can become irregular even at low velocity, high viscosity and in a small tank. The fluid motion is excited in a broad range of spatial and temporal scales, and we observe an increase in the flow resistance by a factor of about twenty. Although the Reynolds number may be arbitrarily low, the observed flow has all the main features of developed turbulence.

A comparable state of turbulent flow for a newtonian fluid in a pipe would have a Reynolds number as high as \(10^5\) (refs 1, 2). The low Reynolds number or ‘elastic’ turbulence that we observe is accompanied by significant stretching of the polymer molecules, resulting in an increase in the elastic stresses of up to two orders of magnitude.

Motion of simple, low molecular weight, newtonian fluids is governed by the Navier–Stokes equation\(^4\). This equation has a nonlinear term, which is due to fluid inertia. The ratio between the nonlinearity and viscous dissipation is given by the Reynolds number, \(Re = V L / \nu\), where \(V\) is velocity, \(L\) is characteristic size and \(\nu\) is kinematic viscosity of the fluid. When \(Re\) is high, nonlinear effects are strong and the flow is likely to be turbulent; therefore, turbulence is a paradigm for a strongly nonlinear phenomenon\(^5\).

Solutions of flexible high molecular weight polymers differ from newtonian fluids in many aspects\(^6\). The most notable elastic property of the polymer solutions is that stress does not immediately become zero when the fluid motion stops, but rather decays with some characteristic time, \(\lambda\), which can reach seconds and even minutes. The equation of motion for dilute polymer solutions differs from the Navier–Stokes equation by an additional linear term arising from the elastic stress, \(\tau\) (ref. 3). Because the elastic stress is caused by stretching of the polymer coils, it depends on history of motion and deformation of fluid elements along their flow trajectories. This implies a nonlinear relationship between \(\tau\) and the rate of deformation in a flow\(^7\). The nonlinear mechanical properties of polymer solutions are well manifest in their large extensional viscosity at high rates of extension\(^8\) and in the Weissenberg effect\(^9\). The degree of nonlinearity in the mechanical properties is expressed by the Weissenberg number, \(Wi = \nu \lambda L / \eta\), which is a product of characteristic rate of deformation and the relaxation time, \(\lambda\).

We considered whether the nonlinearity of mechanical properties of a fluid can give rise to turbulent flow when the equation of motion is linear. For a polymer solution this corresponds to a state in which the Weissenberg number is large, while the Reynolds number is small. This situation can be realized if the parameter of

\[ Wi = \nu \lambda L / \eta \]

Figure 1 The experimental set-up. A stationary cylindrical cup with a plain bottom (the lower plate) is concentric with the rotating upper plate, which is attached to the shaft of a commercial rheometer. The radii of the upper and the lower plates are \(R_1 = 38\) mm and \(R_2 = 43.6\) mm, respectively. The liquid is filled until a level of \(10\) mm unless otherwise stated. The upper plate just touches the surface of the liquid. A special cover is used to minimize evaporation of the liquid. We used a solution of 65% saccharose and 1% NaCl in water, viscosity \(\eta = 0.324\) Pa s, as a solvent for the polymer. We added polyacrylamide (\(M_w = 18,000,000\); Polysciences) at a concentration of 80 p.p.m. by weight. The solution viscosity was \(\eta = 0.424\) Pa s at \(\nu = 1\) s\(^{-1}\). The relaxation time, \(\lambda\), estimated from the phase shift between the stress and the shear rate in oscillatory tests, was 3.4 s. The temperature is stabilized at 12°C by circulating water under the steel lower plate. The walls of the cup are transparent which allows Doppler velocimetry measurements by collecting light scattered from the crossing point of two horizontal laser beams. In experiments where the flow has to be viewed from below, the lower plate is made from plexiglass and a mirror tilted by 45° is placed under the lower plate. The flow patterns are then captured by a CCD camera at the side and the temperature is stabilized by circulating air in a closed box.

elastability, \( \text{Wi/Re} = \lambda \nu/L^2 \), is large enough. An important observation concerning the influence of the nonlinear mechanical properties on flow was made about a decade ago, when purely elastic instability was experimentally identified in curvilinear shear flows\(^{6,7} \). This instability occurs at moderate Wi and vanishingly small Re and is driven by the elastic stresses\(^{7,9} \). As a result of the instability, secondary and in general oscillatory vortex flows develop, and flow resistance increases\(^{8-10} \). Flow instabilities in elastic liquids are reviewed in refs 11, 12.

There is no commonly accepted unique definition of turbulent flow\(^2 \), and it is usually identified by its main features\(^2,3 \). Turbulence implies fluid motion in a broad range of spatial and temporal scales, so that many degrees of freedom are excited in the system. A practically important characteristic of turbulent flows is a large increase in the flow resistance compared with an imaginary laminar flow with the same Re. We show how these main features of turbulence appear in a flow of a highly elastic polymer solution at low Reynolds numbers.

For our experiments we chose a swirling flow between two parallel disks (Fig. 1), and a dilute solution of high molecular weight polyacrylamide in a viscous sugar syrup as the working fluid. The curvature ratio was made quite high, \( d/R = 0.263 \), to provide destabilization of the primary shear flow and development of the secondary vortical fluid motion at lower shear rates\(^{11-16} \). (The flow between two plates with small \( d/R \) has been studied previously in the context of the purely elastic instability\(^{20} \)). We mounted the whole flow set-up on top of a commercial viscometer (AR-1000 of TA-instruments) to measure precisely the angular velocity, \( \omega \), of the rotating upper plate and the torque applied to it. In this way we were able to estimate the average shear stress, \( \sigma \), in the polymer solution and to compare it with the stress in the laminar flow, \( \sigma_{\text{lum}} \), with the same applied shear rate. In newtonian fluids the ratio \( \sigma/\sigma_{\text{lum}} \) generally grows with Re as the flow becomes increasingly irregular, and the magnitude of \( \sigma/\sigma_{\text{lum}} \) can be considered as a measure of strength of turbulence and turbulent resistance. In our set-up \( \sigma \) becomes 30% higher than \( \sigma_{\text{lum}} \) at Re = 70, which can be regarded as a point when inertial effects become significant.

The dependence of \( \sigma/\sigma_{\text{lum}} \) on the shear rate, \( \gamma = \omega r/d \), for flow of the polymer solution in the experimental system is shown in Fig. 2 (first curve). At a value of \( \gamma \) of about 1 s\(^{-1} \) (corresponding to \( \text{Wi} = \lambda \nu = 3.5 \)), a sharp transition occurs that appears as a significant increase in the apparent viscosity. The Reynolds number at the transition point is about 0.3, which means that the inertial effects are quite negligible. The transition has pronounced hysteresis,

![Figure 2](https://example.com/fig2.png)

**Figure 2** Stress ratio versus shear rate. We plot the ratio of the average stress, \( \sigma \), measured in the flow, to the stress, \( \sigma_{\text{lum}} \), in the laminar flow with the same boundary conditions as a function of the shear rate, \( \gamma \). The curves 1 and 2 are for the polymer solution flow with \( d = 10 \) mm and 20 mm, respectively. The shear rate was gradually varied in time, very slowly (by about 10% h\(^{-1} \)) in the transition region, and faster below and above it. Thin black lines represent increasing \( \gamma \); thick grey lines represent decreasing \( \gamma \). Curve 3 represents the pure solvent. Mechanical degradation of the polymers was quite small at shear rates below 1.5 s\(^{-1} \) and 1 s\(^{-1} \) for \( d = 10 \) mm and 20 mm, respectively. The dependences of \( \sigma/\sigma_{\text{lum}} \) on \( \gamma \) in those regions were therefore reproducible in consecutive runs within about 1%. Degradation effects became appreciable at higher shear rates, and elasticity typically decreased by up to 10% as a result of the runs shown by the curves 1 and 2.

![Figure 3](https://example.com/fig3.png)

**Figure 3** Two snapshots of the flow at \( \text{Wi} = 13 \), \( \text{Re} = 0.7 \). The flow under the black upper plate is visualized by seeding the fluid with light reflecting flakes (1% of the Kalliroscope liquid). The fluid is illuminated by ambient light. Although the pattern is quite irregular, structures that appear tend to have spiral-like forms. The dark spot in the middle corresponds to the centre of a big persistent toroidal vortex that has dimensions of the whole set-up.

![Figure 4](https://example.com/fig4.png)

**Figure 4** Average Fourier spectra of the brightness profiles. These are taken along the diameter (thin black line) and along the circumference at a radius of 2d (thick grey line). We averaged over a long series of flow pattern snapshots taken in consecutive moments of time. The wavelength is measured in units of \( d \), so that the wavenumber, \( k_b \), of unity corresponds to a length of 2 \( \pi d \). The spectrum taken along the diameter apparently differs from the azimuthal spectrum by a series of broad peaks. This may be a manifestation of the fact that the flow is not completely structureless and homogeneous along the radial direction (see Fig. 3). The visualization method that we used (Fig. 3) does not provide direct information about the fluid velocity; therefore, the exact value of the exponent in the power law fit, \( A = k_b^{-1} \), has no special meaning.

![Figure 5](https://example.com/fig5.png)

**Figure 5** Power spectra of velocity fluctuations. The data are obtained at different rates, \( \gamma \) in the standard set-up. The fluid velocity was measured by a laser Doppler velocimeter in the centre of the flow. Curves 1–5 correspond to \( \gamma = 1.25, 1.85, 2.7, 4 \) and 5.9 s\(^{-1} \), respectively (all above the transition point \( \gamma = 1 \), Fig. 2). The power, \( P \), of fluctuations is fitted by a power law, \( P \sim f^{-3.5} \), for \( \gamma = 4 \) s\(^{-1} \) over about a decade in frequencies, \( f \).
which is typical for the purely elastic flow instability. The ratio $\sigma/\sigma_{\text{lam}}$ keeps growing with the shear rate, and at the highest $\gamma$ that has been reached the flow resistance is about 12 times larger than is found in laminar flow. In the same range of shear rates, flow of the pure solvent is completely laminar and the ratio $\sigma/\sigma_{\text{lam}}$ is unity within the resolution of the viscometer (about 1%). To make sure that the observed flow phenomena were indeed caused by the solution elasticity, we measured $\sigma$ for a few solutions that had the same polymer concentration but different relaxation times, $\lambda$. The curves of $\sigma/\sigma_{\text{lam}}$ coincided, when plotted against $\lambda$, whereas the Reynolds number turned out to be completely irrelevant (see also ref. 9). The growth of the resistance in the polymer solution flow becomes even larger when the size of the gap is increased (Fig. 2, second curve). At this point the ratio $\sigma/\sigma_{\text{lam}}$ reaches a value of 19. Such growth of the flow resistance is found for Newtonian fluids in the same flow geometry at Re of about $2 \times 10^5$. For flow in a circular pipe this value of $\sigma/\sigma_{\text{lam}}$ is reached at $Re = 10^5$, which is usually considered as a region of rather developed turbulence.

Two representative snapshots of the polymer solution flow above the transition (at $\gamma = 4 \text{s}^{-1}$) are shown in Fig. 3. The flow patterns are very irregular and structures of different sizes appear. This visual impression is confirmed by a more careful analysis. Average Fourier spectra of the brightness profiles along the diameter and along the circumference exhibit power law decay over a decade in the wavenumber domain (Fig. 4).

Characteristic time spectra of velocity fluctuations at different shear rates are shown in Fig. 5. Flow velocity was measured in the horizontal plane in the centre of the set-up, where its average value is zero. As the shear rate is raised the power of fluctuations increases and characteristic frequencies become higher, but the general form of the spectra remains very much the same. In particular, just as for the spatial spectra in Fig. 4, there is a region of a power law decay, which spans about a decade in frequencies. This power law dependence in the broad ranges of spatial and temporal frequencies actually means that the fluid motion is excited at all those spatial and temporal scales. Spectra of radial and azimuthal velocities taken at different points with non-zero average flow had the same general appearance and close values of exponents in the power law decay range.

In summary, we conclude that the flow of the elastic polymer solution at sufficiently high Wi has all the main features of developed turbulence. By the strength of the turbulent resistance, and by the span of scales in space and time, where the fluid motion is excited, the observed flow can be compared with turbulence in a Newtonian fluid in a pipe at Re of about $10^5$. This apparently turbulent flow arises solely because of the nonlinear mechanical properties of the elastic polymer solution. We therefore call the phenomenon elastic turbulence, in contrast to the usual inertial turbulence which is observed in Newtonian fluids at high Re. (The name ‘elastic turbulence’ has been used before for designation of apparently disordered flows in polymeric liquids 13. No attempt has been ever made, however, to characterize those flows quantitatively.)

Elastic turbulence has many features that are in sharp contradiction to the concepts of Newtonian fluid mechanics. On the one hand, velocity required for excitation of inertial turbulence in a Newtonian fluid is proportional to the fluid viscosity. On the other hand, the polymer relaxation time, $\lambda$, usually grows proportionally to the viscosity. Because at constant $d/R$ the transition to elastic turbulence occurs at a certain value of the Weissenberg number, $Wi = \dot{\gamma} \lambda / \eta$, one can excite turbulence at lower velocities by choosing more viscous polymer solutions. Indeed, using a solution of polymers in a very viscous sugar syrup, we observed transition to the elastic turbulence at a rotation rate of 0.05 s$^{-1}$ (corresponding to Re = $10^3$). Further, in an elastic polymer solution, the scale of time, $\lambda$, does not depend on the size of the system; therefore as long as the ratio $d/R$ is preserved, transition to turbulence should occur at the same $\omega$, and the dependence of $\sigma/\sigma_{\text{lam}}$ on $\gamma$ should not change with the size of the system. We repeated the measurements of $\sigma$ in a small set-up having all the dimensions reduced by a factor of 4 as compared with the standard system. The dependence of $\sigma/\sigma_{\text{lam}}$ on $\gamma$ was found to be the same (data not shown) as in Fig. 2, whereas the characteristic velocities and the Reynolds numbers were lower by factors of 4 and 16, respectively. We therefore believe that by using polymer solutions with sufficiently high elasticity we can excite turbulent motion at arbitrary low velocities and in arbitrary small tanks. (The size of the tank still has to be large compared with the size of the polymer coils.)

An important question about the elastic turbulence is where the turbulent resistance comes from. In the inertial turbulence the origin of the large resistance is the Reynolds stress, which is connected with high kinetic energy of the turbulent motion and takes a major part in the momentum transfer in the flow. Elastic turbulence occurs at low Reynolds numbers. From our velocity measurements in the standard set-up, contribution of the Reynolds stress to the flow resistance could be estimated as being less than 0.5%. The contribution of the viscous shear stress of the Newtonian solvent, averaged across the fluid layer, is always the same as in laminar flow and cannot change. Thus, the whole increase in the flow resistance should be due to the elastic stress. The data shown in Fig. 2 (second curve) imply that the polymer contribution to the stress increases by a factor of up to 65, as compared with laminar flow with the same average shear rate. This suggestion agrees very well with our measurements of relaxation of the shear stress after the fluid motion is stopped. The elastic part, $\tau$, of the whole stress is identified by its slow relaxation with a characteristic time of the order $\lambda$. In elastic turbulence this slowly relaxing part can become two orders of magnitude larger than in laminar flow with the same shear rate. This major growth of the elastic stress should be connected with vast extension of the polymer molecules in the turbulent flow.

Thus, elastic turbulence apparently develops as follows. The polymer molecules are stretched in the primary shear flow, which makes it unstable and causes irregular secondary flow. This flow acts back on the polymer molecules, stretching them further and becoming increasingly turbulent, until a kind of saturated dynamic state is reached. The density of the elastic energy of the stretched polymers can be estimated as $Wi \tau/2$, and should therefore increase in the elastic turbulent flow by about the same factor as the elastic stress, $\tau$, while the kinetic energy remains small.

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