Spinning of a molten threadline

Steady-state isothermal viscous flows

Jet equations and shape

M.A. Matovich and J.R.A. Pearson (1969)

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Summer Reading Group

The authors

J.R.A. Pearson

- Area of research:
  polymer melt processing
  (mechanics, computational analysis)
- Worked/Work at Schlumberger, Cambridge, UK
- Honorary professor at University of Wales Aberystwyth
- Member of the editor board of JNNF

M.A. Matovich

- PhD at Cambridge, UK, and apparently post-doc with Pearson.
- Worked/Works for Shell, Emeryville, CA, on gas combustion

Motivation of the problem

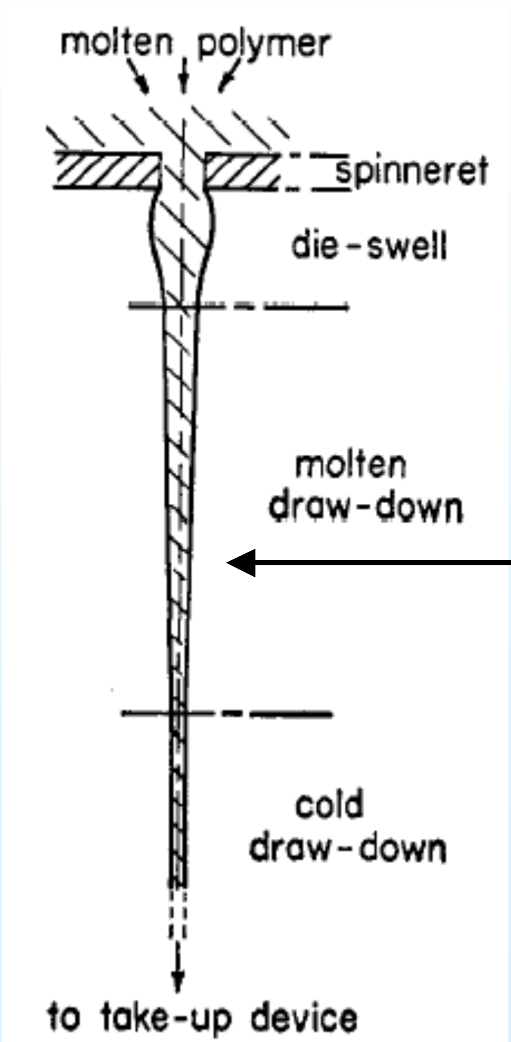
- The main application is the understanding of **fiber-drawing process of polymer melts**, e.g. Dacron (poly(ethylene therephtalate)), polypropylene, Nylon (polyamide). VERY relevant industrially.

- Valuable information that we want with respect to the boundary conditions: **radius**, **extension rate/jet shape** (which has a strong influence on fiber properties)

- Also relevant is the stability of the jet (Pearson & Matovich 1969, *Spinning a Molten Threadline, Stability*), the stable operating space, and what parameters affect **spinnability** (=stability far from orifice).

- It can be extended to a lot of problems : non-isothermal, planar extrusion, steady jet on a planar surface...

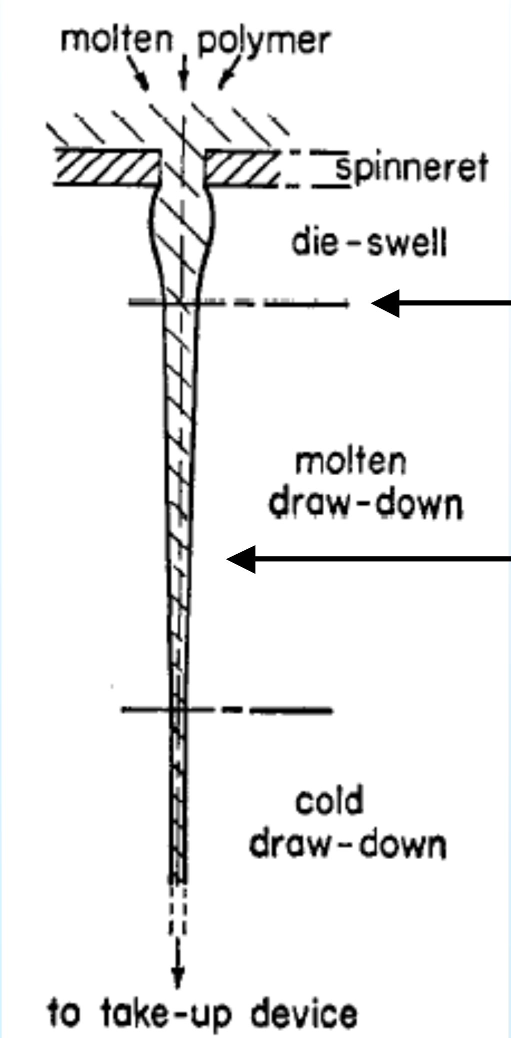
Definition of the problem



We consider
only that part

Variables

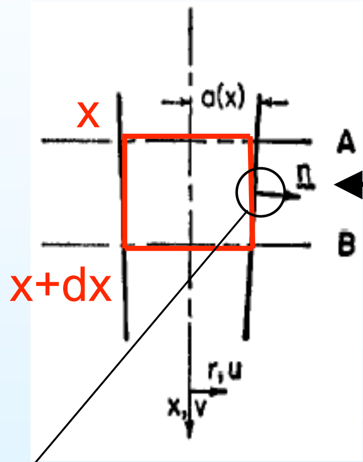
Definition of the problem



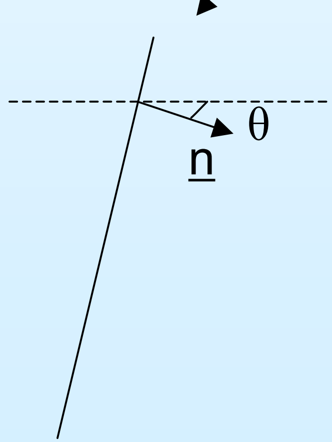
X=0

We consider only that part

Variables



Normal vector $\underline{n}$



$\sin(\theta) \approx -a'$

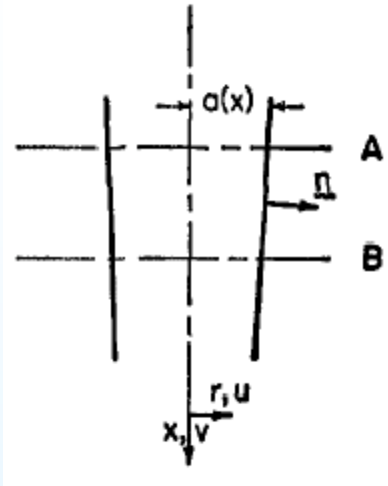
$\cos(\theta) \approx 1 + o(a')$

Therefore

$\underline{n} = (\underline{r} - a'\underline{x})/(1+a'^2)^{1/2}$ (7)

(beware typo in paper)

Flow equations



- <u>Continuity</u>

  Div(<u>u</u>)=0 gives, in cylindrical coordinates:

$$\frac{\partial v}{\partial x} + \frac{1}{r}\frac{\partial (ru)}{\partial r} = 0 \qquad (1)$$

- <u>Conservation of momentum</u>

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = \rho f_i + \frac{\partial \tau_{ij}}{\partial x_j}$$

in steady state and in cylindrical, gives:

r-Momentum

$$\rho\left(u\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial x}\right) = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) - \frac{1}{r}\tau_{\theta\theta} + \frac{\partial \tau_{rx}}{\partial x} \qquad (2)$$

x-Momentum

$$\rho\left(u\frac{\partial v}{\partial r} + v\frac{\partial v}{\partial x}\right) = \rho g + \frac{1}{r}\frac{\partial (r\tau_{rr})}{\partial r} + \frac{\partial \tau_{xx}}{\partial x} \qquad (3)$$

Here τ is the total stress tensor, (usually written σ, with τ being the deviatoric stress)

Boundary conditions

- Radial BC (at r=a(x)):
  - Kinematic : the surface is a streamline, thus

$$v a' = u \tag{4}$$

  - Stress: free surface, no shear stress

    The Laplace pressure difference is σC, where C is the sum of the 2 curvatures : 1/a and $-a''^{1/2}/(1+a'^2)^{3/2}$

$$\sigma\left(\frac{1}{R} - \frac{1}{a}\right) n_r = \tau_{rr} n_r + \tau_{rx} n_x \tag{5}$$

$$\sigma\left(\frac{1}{R} - \frac{1}{a}\right) n_x = \tau_{rx} n_r + \tau_{xx} n_x \tag{6}$$

Boundary conditions

- Upstream and/or downstream BCs:
  - imposed initial flow rate
    $$\left.\begin{aligned} a &= a_0 \\ v &= v_0,\ \text{const.} \end{aligned}\right\} \quad \text{at } x = 0 \qquad (9)$$
  - plus one of the following:
    - Imposed final speed
      $$v = v_1,\ \text{const.,} \quad \text{at } x = l \qquad (10\text{i})$$
    - Imposed final force
      $$\left\{\begin{aligned} &2\pi \int_0^{a_0} \tau_{xx} r\,dr = F_{t0}, \quad \text{at } x = 0 \qquad (10\text{ii}) \\ &2\pi \int_0^{a} \tau_{xx} r\,dr = F_{t1}, \quad \text{at } x = l \qquad (10\text{iii}) \end{aligned}\right.$$

Approximation scheme

- Development in power of a', which is <<1

$$v = v^{(0)}(x) + v^{(1)}(r,x) + v^{(2)}(r,x) + \ldots$$

$$u = u^{(1)}(r,x) + u^{(2)}(r,x) + \ldots$$

$$p = p^{(0)}(x) + p^{(1)}(r,x) + p^{(2)}(r,x) + \ldots$$

$$\tau_{xx} = \tau_{xx}{}^{(0)}(x) + \tau_{xx}{}^{(1)}(r,x) + \tau_{xx}{}^{(2)}(r,x) + \ldots \quad (12)$$

$$\tau_{rr} = \tau_{rr}{}^{(0)}(x) + \tau_{rr}{}^{(1)}(r,x) + \tau_{rr}{}^{(2)}(r,x) + \ldots$$

$$\tau_{\theta\theta} = \tau_{\theta\theta}{}^{(0)}(x) + \tau_{\theta\theta}{}^{(1)}(r,x) + \tau_{\theta\theta}{}^{(2)}(r,x) + \ldots$$

$$\tau_{rx} = \tau_{rx}{}^{(1)}(r,x) + \tau_{rx}{}^{(2)}(r,x) + \ldots$$

$$a = a^{(0)}(x) + a^{(1)}(x) + a^{(2)}(x) + \ldots \quad (13)$$

- Equations (22) through (30) are a proof of self-consistency, and a guide towards computing higher-order terms.

Approximation scheme (cont'd)

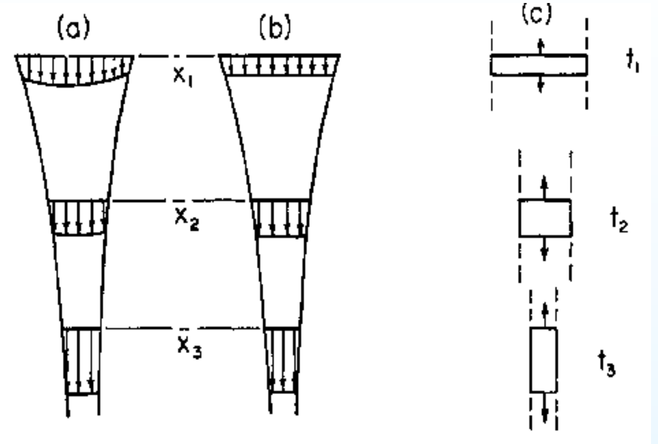


Figure 3. Rheological characterization of fluid

- Thin jet approximation : 0-order term are independent of r

- 1st-order momentum equation

$$\rho v^{(0)} v^{(0)\prime} = \rho g + \tau_{xx}{}^{(0)\prime} + \frac{2a^{(0)\prime}}{a^{(0)}}\left[\tau_{xx}{}^{(0)} + \frac{\sigma}{a^{(0)}}\right] \qquad (20)$$

-The trick to easily derive (20) from (3) is to use the *integral* form, and retain only 1-order terms (top of page 515). That way, a' shows up only in the change of area, and 1^{st} and higher order terms of the expansion cancel out.

- a and a' are converted into v and v' using the conservation of flow rate (11)

Approximation scheme (cont'd)

Scaling of the different terms with a parameter ε

$$a_\epsilon(x) = \epsilon^\alpha a(x\epsilon^{-\gamma})$$

$$v_\epsilon(x) = \epsilon^\beta v(x\epsilon^{-\gamma}) \qquad (31)$$

$$v_\epsilon'(x) = \epsilon^{\beta-\gamma} v'(x\epsilon^{-\gamma})$$

$$a_\epsilon'(x) = \epsilon^{\alpha-\gamma} a'(x\epsilon^{-\gamma}) \qquad (32)$$

Analogous solution:

terms must keep the same scaling even for a'→ 0 $\alpha > \gamma;\ \epsilon \to 0$

From this, they deduce the scaling of the parameters (33):

$$\sigma_\epsilon = \epsilon^\alpha \sigma,$$

$$\rho_\epsilon = \epsilon^{-2\beta} \rho$$

$$(\rho g)_\epsilon = \epsilon^{-\beta} \rho g,$$

Relationship between α and β given by a' scaling and β = γ

Solutions

One need to provide a constitutive equation, then plug it into (20)

- Newtonian case
  - Constitutive equation $\boldsymbol{\tau} = -p\mathbf{I} + \eta_0 \mathbf{e} \qquad (21)$
  - Momentum equation

$$\underbrace{\rho v v'}_{\text{Inertia}} = \underbrace{\rho g}_{\text{Gravity}} \underbrace{- 3\eta_0 \frac{(v')^2}{v} + 3\eta_0 v''}_{\text{Viscosity}} \underbrace{- \sigma \pi^{1/2} \frac{v'}{2Q^{1/2} v^{1/2}}}_{\text{Surface tension}} \qquad (34)$$

  - The relative importance of the different terms is given by
    - Viscosity: 1
    - Inertia: Reynolds number Re
    - Gravity: Froude number Fr, or gravity number B=Re/Fr
    - Surface tension: Weber number We, or capillary number 1/Ca=Re/We

$$\mathrm{Re} = \frac{\rho L_c V_c}{3\eta_0}, \quad \mathrm{Fr} = \frac{V_c^2}{g L_c},$$

$$\mathrm{We} = 2\rho (Q/\pi)^{1/2} (V_c^{3/2}/\sigma) = \frac{2a_0 V_c^2 \rho}{\sigma}$$

Solutions (cont'd)

- Newtonian case
  - Viscous-only solution (Re, Re/Fr, Re/We <<1)

$$v(x) = v_0 \exp\,(x/L_c) \qquad (37)$$

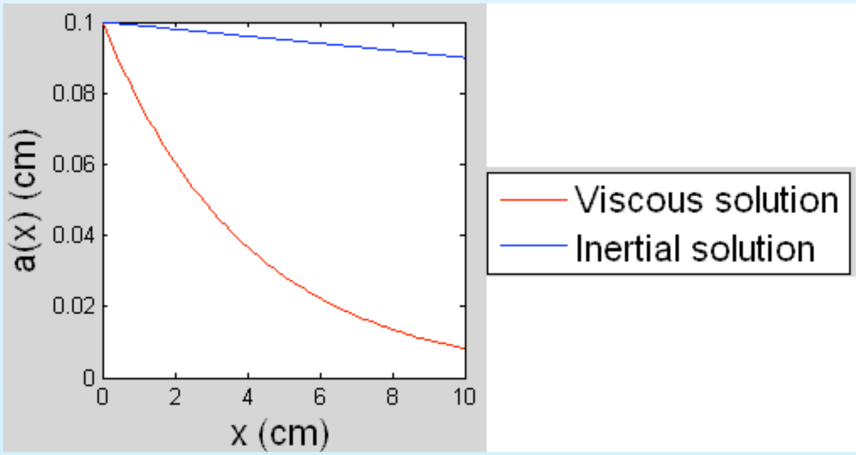
$$L_c = \begin{cases} l/\ln\,(v_1/v_0) & (38\text{i}) \\ \text{or } \eta_0 Q/F_{t_0} & (38\text{ii}) \\ \text{or } \eta_0 Q/F_{t_1} & (38\text{iii}) \end{cases}$$

Depending on the BC

  - Visco-inertial solution (Re ≈ 1, Re/Fr, Re/We <<1)

$$v(x) = c_1\left[c_2 \exp\,(-c_1 x) - \tfrac{1}{3}(\rho/\eta_0)\right]^{-1} \qquad (39)$$

Sketch of the solutions for a_0=1mm, and arbitrary constants



Solutions (cont'd)

Newtonian case (cont'd)

– Visco-gravitational solution (Re/Fr ≈ 1,

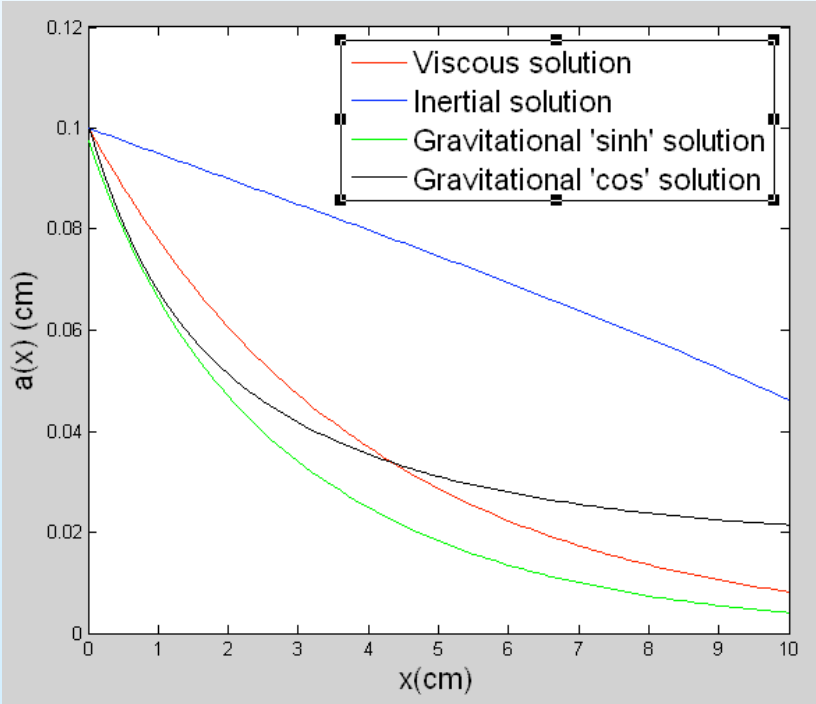
$$v(x) = (2\rho g/3\eta_0 c_1)\sinh^2\{\tfrac{1}{2}c_1^{1/2}(x + c_2)\} \qquad (40)$$

Comes from Trouton, (1906). Determining the constants c_1 and c_2 is easier said than done…

Ribe (2004) gives another solution, for the BC (i), which has a small range of application: $v = v_1 \cos^2(\sqrt{\rho g Q / v_1}(x + x_1))$

– Viscosity and surface tension (Re/We ≈ 1, Re, Re/Fr <<1)

$$v(x) = \left[\frac{c_2}{c_1}\exp\left(\tfrac{1}{2}c_1 x\right) + \frac{\sigma\pi^{1/2}}{3\pi_0 Q^{1/2} c_1}\right]^2 \qquad (41)$$



– Inviscid solutions (Re, Re/Fr, Re/We >>1) are not of concern here. They can be found for example in *The Mechanics of Liquid Jets*, by J.N. Anno.

Solutions (cont'd)

- <u>Non-Newtonian case: a lot of models are available</u>
  - A simple one is the inelastic fluid model: the newtonian viscosity is replaced by a Trouton viscosity

$$\tau_{xx} = -(\sigma/a) + \eta_T(v')v' \qquad (44)$$

This gives a momentum equation of the form (45):

$$\rho v v' = \rho g - 3\eta_T \frac{(v')^2}{v} + 3(\eta_T + v'\eta_T')v'' - \sigma\pi^{1/2}\frac{v'}{2Q^{1/2}v^{1/2}}$$

Here again, different models for the viscosity. The simplest is the power-law model:

$$\eta_T = \eta_p (v')^{q-1} \qquad (46)$$

The solution is easy for viscous-only case:

$$v(x) = \{v_0^m + (v_1^m - v_0^m)(x/L)\}^{1/m}$$

$$m = (q-1)/q \qquad (47)$$

As expected, shear-thinning hinders spinnability.

Solutions (cont'd)

- Non-Newtonian case: a lot of models are available
  - A second step towards difficulty is the second-order fluids model:

$$\boldsymbol{\tau} = -p\mathbf{I} + \eta_0 \mathbf{d} - \eta_1 \overset{\nabla}{\mathbf{d}} + \eta_2 \mathbf{d}^2, \qquad (52)$$

    This leads to a third-order differential equation for the conservation of momentum:

$$\rho v v' = \rho g + 3\eta_0\left[v''' - \frac{(v')^2}{v}\right] + 3\eta_0\tau_c\left\{\xi\left[2v'v'' - \frac{(v')^3}{v}\right] - vv'''\right\} + \frac{\sigma a'}{a^2} \qquad (53)$$

  - To solve it, they use an expansion in powers of a Deborah number Δ
    In dimensionless form, $\psi = \psi_0 + \Delta\psi_1 + \Delta^2\psi_2 + \ldots + \Delta^n\psi_n + 0(\Delta^n) \qquad (57)$

    Then, every order gets its own equation (and needs its own 2 BCs…)

    0-order:

$$\psi_0\psi_0'' - (\psi_0')^2 = 0 \qquad (58.0)$$

    1st-order:

$$\psi_0\psi_1'' - 2\psi_0'\psi_1' + \psi_0''\psi_1 = \xi\{(\psi_0')^3 - 2\psi_0\psi_0'\psi_0''\} + \psi_0^2\psi''' \qquad (58.1)$$

    nth-order

$$\psi_0\psi_n'' - 2\psi_0\psi_n' + \psi_0''\psi_n = f_n(\psi_{n-1}, \ldots, \psi_0) \qquad (58.n)$$

Solutions (cont'd)

- Non-Newtonian case

They give the solutions for the first two orders

0-order:

$$\psi_0\psi_0'' - (\psi_0')^2 = 0 \qquad (58.0)$$

gives

$$\psi_0 = C_1 e^{C_2 X} \qquad (59)$$

1st-order:

$$\psi_0\psi_1'' - 2\psi_0'\psi_1' + \psi_0''\psi_1 = \xi\{(\psi_0')^3 - 2\psi_0\psi_0'\psi_0''\} + \psi_0^2\psi''' \qquad (58.1)$$

gives

$$\psi_1 = \begin{cases} (1-\xi)\left\{e^{2X} - e^X - \dfrac{(v_1 - v_0)Xe^X}{v_0 \ln (v_1/v_0)}\right\} & (64\text{i}) \\ (1-\xi)\{e^{2X} - (1+X)e^X\} & (64\text{ii}) \\ (1-\xi)\{e^{2X} - e^X - e^2 X e^X\} & (64\text{iii}) \end{cases}$$

Depending on the BC

(65) through (71) discuss the validity of the solution, depending on the BCs (the perturbation method loses ground for Δ too large) and give another derivation route.

Extension to Nonisothermal flows

One needs:

- An equation of state (which can be T-dependent)
- To include temperature convection in flow equation

$$\rho C_p\left(u\frac{\partial T}{\partial r}+v\frac{\partial T}{\partial x}\right)=k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial x^2}\right]+$$

$$\left[\tau_{rr}\frac{\partial u}{\partial r}+\tau_{\theta\theta}\frac{u}{r}+\tau_{xx}\frac{\partial v}{\partial x}+\tau_{rx}\left(\frac{\partial v}{\partial r}+\frac{\partial u}{\partial x}\right)\right] \quad (72)$$

- One radial and two axial boundary conditions for temperature. The most obvious is $T=T_1$ for the melt reservoir, $T=T_0$ for the ambient air, and a flux at the interface proportional to $T-T_0$ ((73) to (75)).

Extension to Fiber drawing and Film casting stability

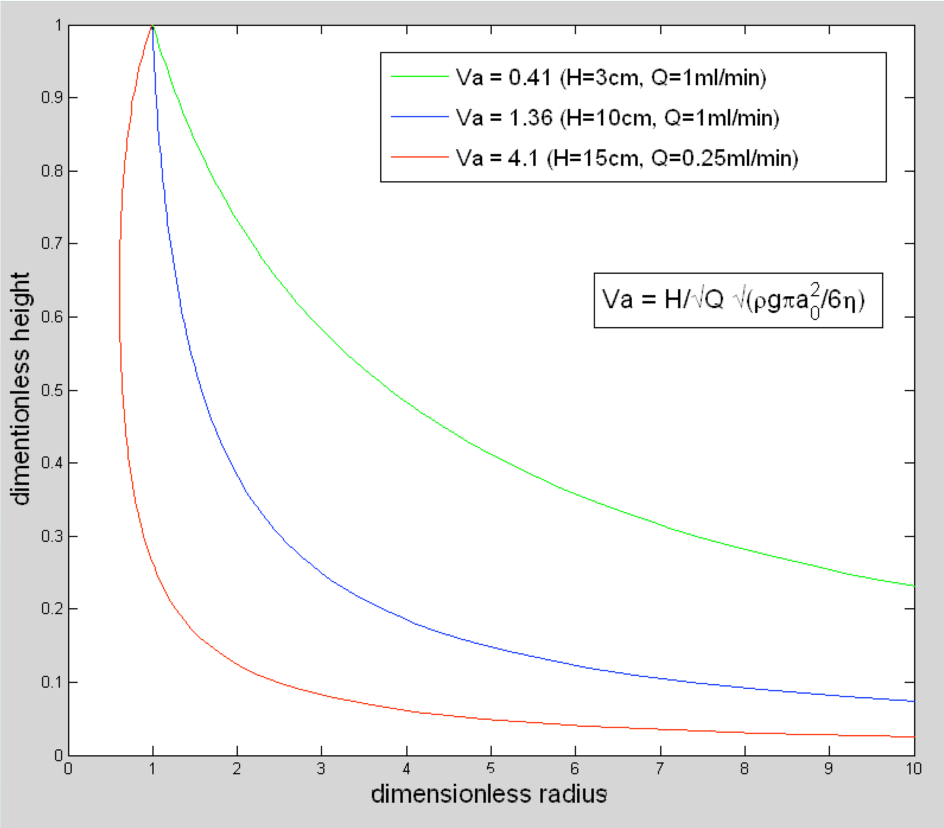
- Jet stability :
  Pearson & Matovich 1969, *Spinning a Molten Threadline, Stability.* They take in account different causes of instability : radius or speed varying at the origin, speed or tension varying at the wind-up (but they don't take in account extension thickenning, which should play a role in stabilizing…).

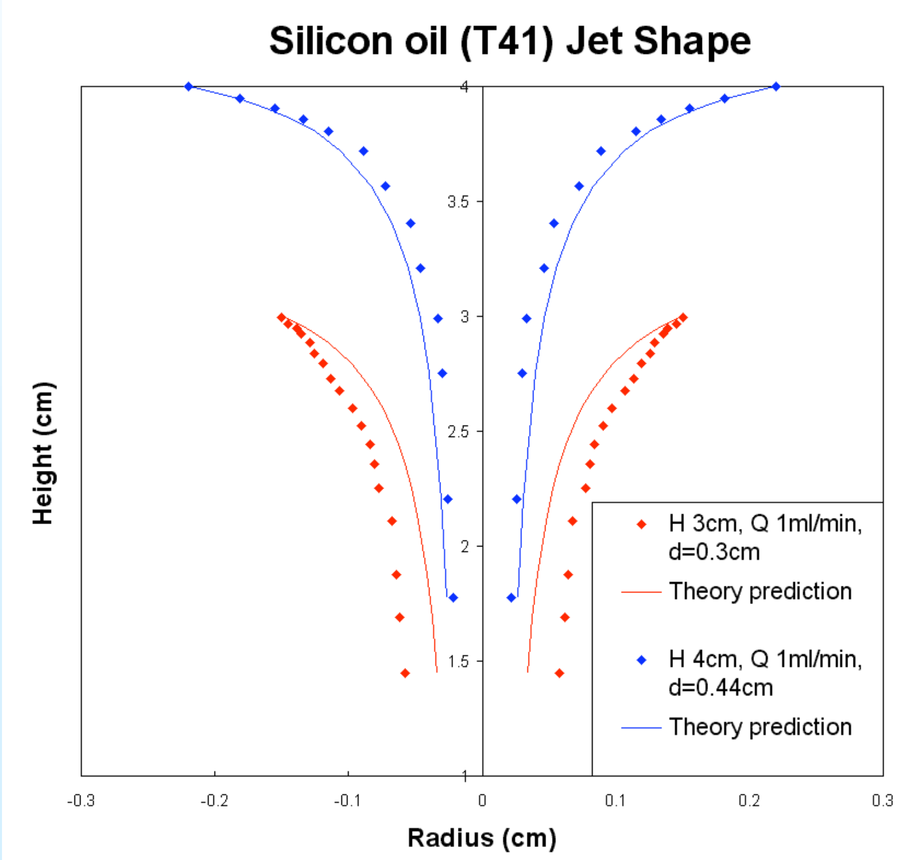
- Film casting :
  Yeow (J. Fluid Mech., 1974). They problem is no longer axisymmetric.

Extension to jet on a plate

- <u>Steady jet</u>:
  Cruikshank and Munson (1982). "v=0 at the plate" boundary condition.

- <u>Coiling jet</u>:
  The speed at the plate is **non-zero, non-imposed: we lose a boundary condition**. Ribe (2004) gives a scaling argument for the visco-gravitational case.





Three different problems

- Matovitch & Pearson : Drawn fiber, ie final speed or force imposed.
- Cruikshank & Munson : Steady jet on a plate, ie speed = 0 at the plate.
- Our problem : Non-steady jet on a plate, ie, non-imposed, non-zero speed at the plate.