The solid mechanics view on continuum elastic-viscoplastic deformation

Shawn Chester, David Henann,
Vikas Srivastava

M.I.T.

Outline

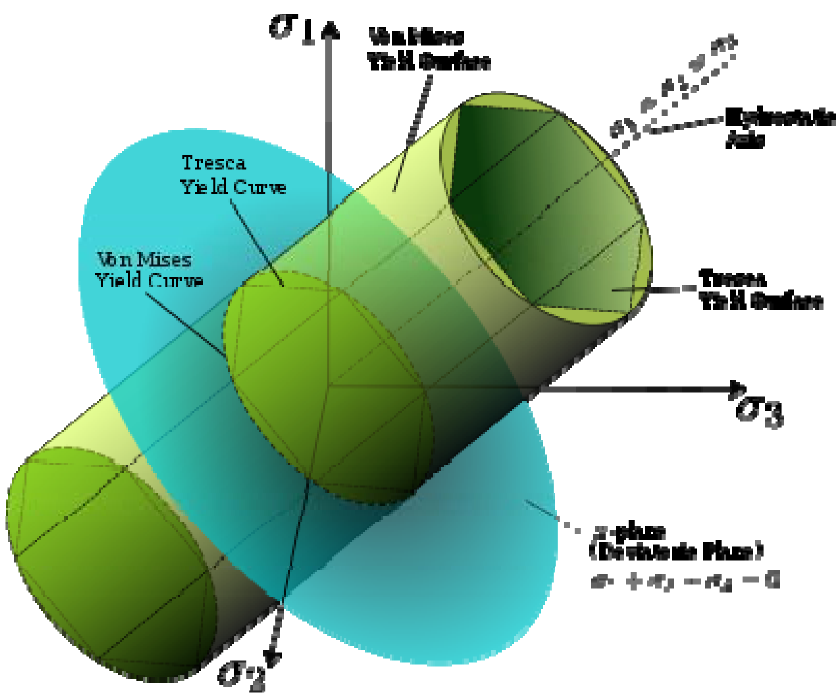
- Tresca vs. Mises yield criteria
- Continuum Plasticity
  - Kinematics (3D, 1D large and small)
  - Constitutive theory
  - Specialization to a Bingham material
- 2D vs. 3D
- 1D implementation

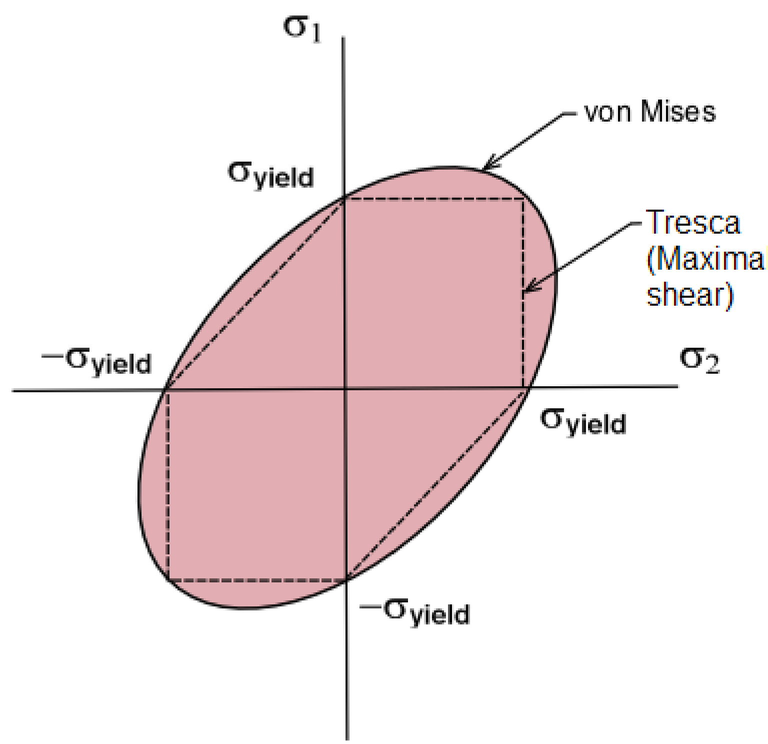
Tresca vs. Mises

- **Mises:** Maximum distortional energy

$$\bar{\sigma} = \sqrt{\frac{1}{2}\left((\sigma_{11}-\sigma_{22})^2+(\sigma_{11}-\sigma_{33})^2+(\sigma_{22}-\sigma_{33})^2\right)+3\left(\sigma_{12}^2+\sigma_{13}^2+\sigma_{32}^2\right)}$$

- **Tresca:** Maximum shear stress $\tau = \frac{\sigma_1 - \sigma_3}{2}$





1D small-deformation elastic-viscoplastic

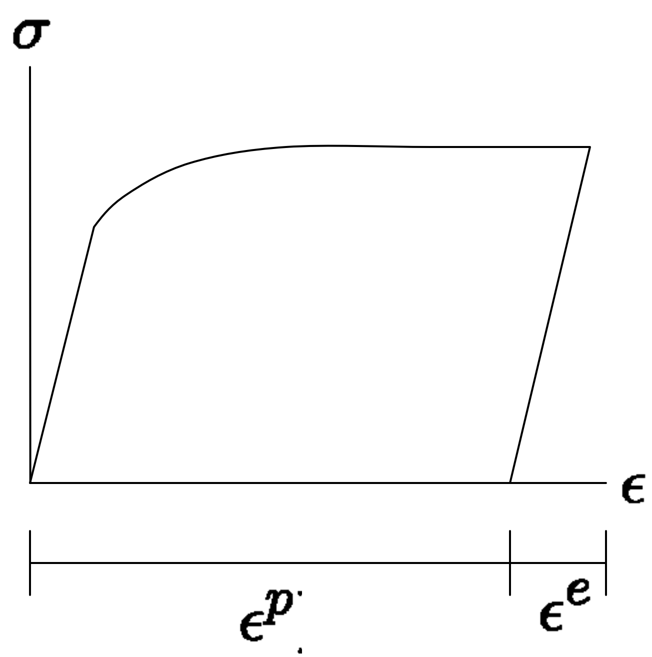
Elastic-plastic decomposition of ϵ

$$\epsilon = \epsilon^e + \epsilon^p,$$

$$\dot{\epsilon} = \dot{\epsilon}^e + \dot{\epsilon}^p,$$

Elastic stress-strain relation

$$\sigma = E\,\epsilon^e = E\,(\epsilon - \epsilon^p).$$



Flow rule: Let

$$n^p = \frac{\dot{\epsilon}^p}{|\dot{\epsilon}^p|}$$

denote the plastic flow direction, and

$$\dot{\bar{\epsilon}}^p = |\dot{\epsilon}^p| \geq 0,$$

the equivalent tensile plastic strain rate. We assume that plastic flow occurs in the direction of the stress:

$$n^p = \mathrm{sign}(\sigma).$$

Hence,

$$\dot{\epsilon}^p = \dot{\bar{\epsilon}}^p n^p, \quad \text{where} \quad n^p = \mathrm{sign}(\sigma) \quad \text{and} \quad \dot{\bar{\epsilon}}^p \geq 0.$$

Equivalent plastic tensile strain rate $\dot{\bar{\epsilon}}^p$

$$\dot{\bar{\epsilon}}^p = f(|\sigma|) > 0 \quad \text{with} \quad \dot{\bar{\epsilon}}^p = 0 \;\; \text{when} \;\; |\sigma| = 0.$$

3D small-deformation elastic-viscoplastic

Kinematics: $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^p, \quad \text{with} \quad \mathrm{tr}\,\boldsymbol{\epsilon}^{\mathrm{p}} = 0,$

Equation for stress:

$$\boldsymbol{\sigma} = 2G\boldsymbol{\epsilon}^e + (K - \frac{2}{3}G)(\mathrm{tr}\,\boldsymbol{\epsilon}^e)\mathbf{1}, \quad \bar{\sigma} \stackrel{\mathrm{def}}{=} \sqrt{\frac{3}{2}}|\boldsymbol{\sigma}_0|,$$

Flow rule:

$$\dot{\boldsymbol{\epsilon}}^p = \frac{3}{2}\dot{\bar{\epsilon}}^p\frac{\boldsymbol{\sigma}_0}{\bar{\sigma}}, \quad \dot{\bar{\epsilon}}^p \stackrel{\mathrm{def}}{=} \sqrt{\frac{2}{3}}|\dot{\boldsymbol{\epsilon}}^p|,$$

Equivalent plastic tensile strain rate $\dot{\bar{\epsilon}}^p$

$$\dot{\bar{\epsilon}}^p = f(\bar{\sigma}) \quad \text{with} \quad \dot{\bar{\epsilon}}^p = 0 \ \text{when} \ \bar{\sigma} = 0.$$

Large-deformation kinematics

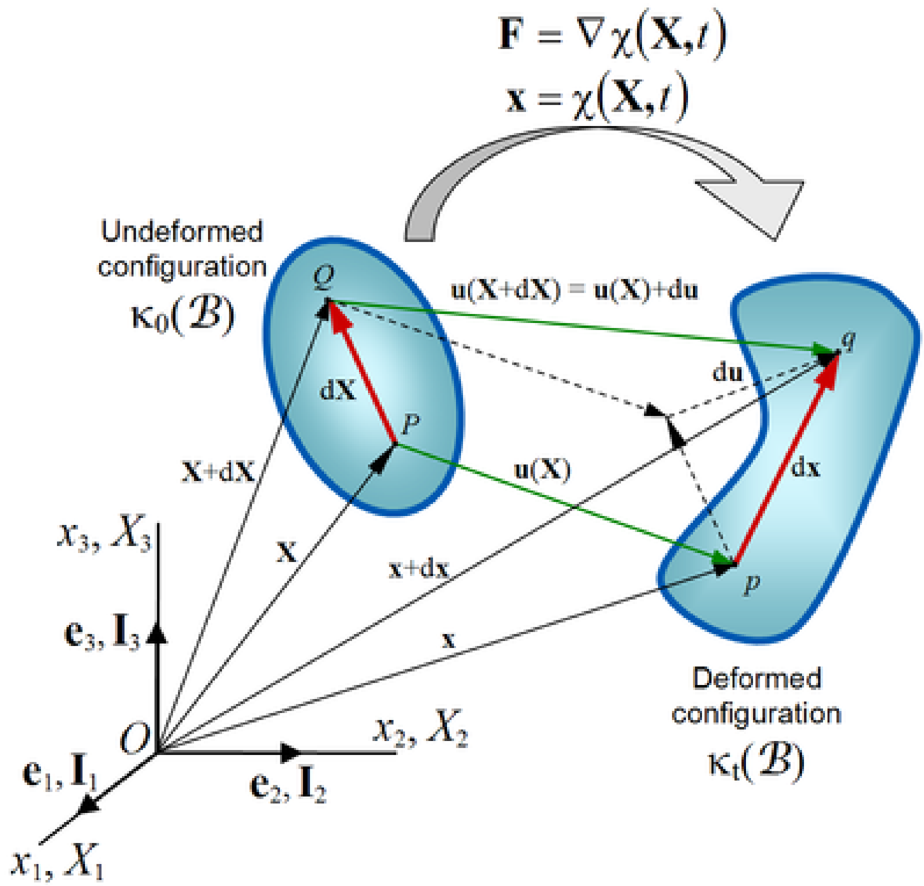
- Deformation Gradient

$$F_{ij} = \frac{\partial x_i}{\partial X_j}$$

- Velocity Gradient

$$\mathbf{L} = \text{grad}\mathbf{v} = \dot{\mathbf{F}}\mathbf{F}^{-1}$$

$$\mathbf{L} = \mathbf{D} + \mathbf{W}$$



Kinematical decomposition: Motivation

- Irreversible part of deformation:

We assume that irreversible flow is due to the flow of “defects” through the material structure

- Reversible part of the deformation:

We assume that reversible deformation is accommodated by stretch and rotation of the structure

Kinematical decompositions

- Small Strain elastic-plastic

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^p \qquad \epsilon_{ij} = \epsilon_{ij}^e + \epsilon_{ij}^p$$

- Large Strain elastic-plastic

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p \qquad F_{ij} = F_{ik}^e F_{kj}^p$$

- In most typical cases, we assume plastic flow is incompressible

$$\mathrm{tr}\,\boldsymbol{\epsilon}^p = 0, \qquad \det \mathbf{F}^p = 1$$

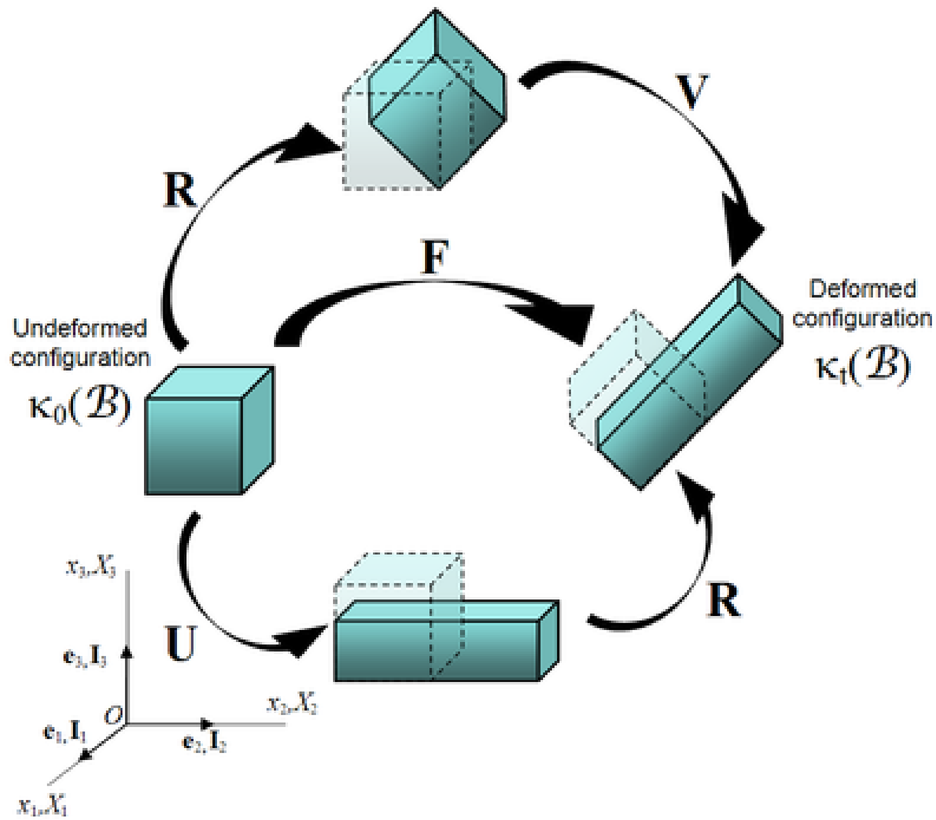
Kinematics

- Polar decomposition

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$$

- Spectral decomposition

$$\mathbf{F} = \sum_{i=1}^{3} \lambda_i \mathbf{l}_i \otimes \mathbf{r}_i$$



Basic Laws

- Cauchy stress, $\mathbf{T}$, $\boldsymbol{\sigma}$

- Conservation of linear momentum

$$\mathrm{div}\mathbf{T} + \mathbf{b} = \rho\ddot{\mathbf{u}}$$

- Conservation of angular momentum

$$\mathbf{T} = \mathbf{T}^{\mathsf{T}}$$

3D large-deformation elastic-viscoplastic

Kinematics: $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$, with $\det \mathbf{F}^{\mathrm{p}} = 1$,

Mandel stress (driving stress for plastic flow):

$$\mathbf{M}^e = 2G\mathbf{E}^e + (K - \frac{2}{3}G)(\mathrm{tr}\,\mathbf{E}^e)\mathbf{1}, \quad \bar{\sigma} \stackrel{\mathrm{def}}{=} \sqrt{\frac{3}{2}}|\mathbf{M}_0^e|.$$

Cauchy stress: $\mathbf{T} = J^{e-1}\mathbf{R}^e\mathbf{M}^e\mathbf{R}^{e\top}$, $J^e = \det \mathbf{F}^e$.

3D large-deformation elastic-viscoplastic

Flow rule:

$$\dot{\mathbf{F}}^p = \mathbf{D}^p\mathbf{F}^p, \qquad \mathbf{D}^p = \frac{3}{2}\dot{\bar{\epsilon}}^p\frac{\mathbf{M}_0^e}{\bar{\sigma}}, \qquad \dot{\bar{\epsilon}}^p \stackrel{\text{def}}{=} \sqrt{\frac{2}{3}}|\mathbf{D}^p|.$$

Equivalent plastic tensile strain rate $\dot{\bar{\epsilon}}^p$

$$\dot{\bar{\epsilon}}^p = f(\bar{\sigma}) \quad \text{with} \quad \dot{\bar{\epsilon}}^p = 0 \ \text{ when } \ \bar{\sigma} = 0.$$

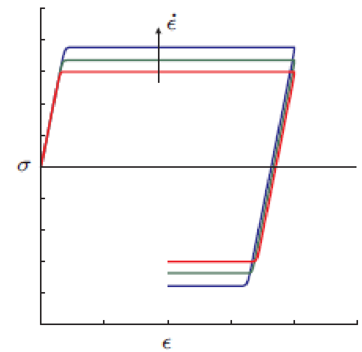
Flow rule

Equivalent tensile plastic strain rate $\dot{\bar{\epsilon}}^p$ needs an constitutive equation. A simple **power-law** function:

$$\dot{\bar{\epsilon}}^p = \dot{\epsilon}_0 \left(\frac{\bar{\sigma}}{S} \right)^{(1/m)},$$

where $\bar{\sigma} \stackrel{\text{def}}{=} |\sigma|$. The inverted form of power-law:

$$\bar{\sigma} = S \left(\frac{\dot{\bar{\epsilon}}^p}{\dot{\epsilon}_0} \right)^m .$$



For $m = 1$, we have **Newtonian viscosity**:

$$\bar{\sigma} = S \left(\frac{\dot{\bar{\epsilon}}^p}{\dot{\epsilon}_0} \right) \rightarrow \eta \stackrel{\text{def}}{=} \frac{S}{\dot{\epsilon}_0} = \frac{\bar{\sigma}}{\dot{\bar{\epsilon}}^p} .$$

Flow rule

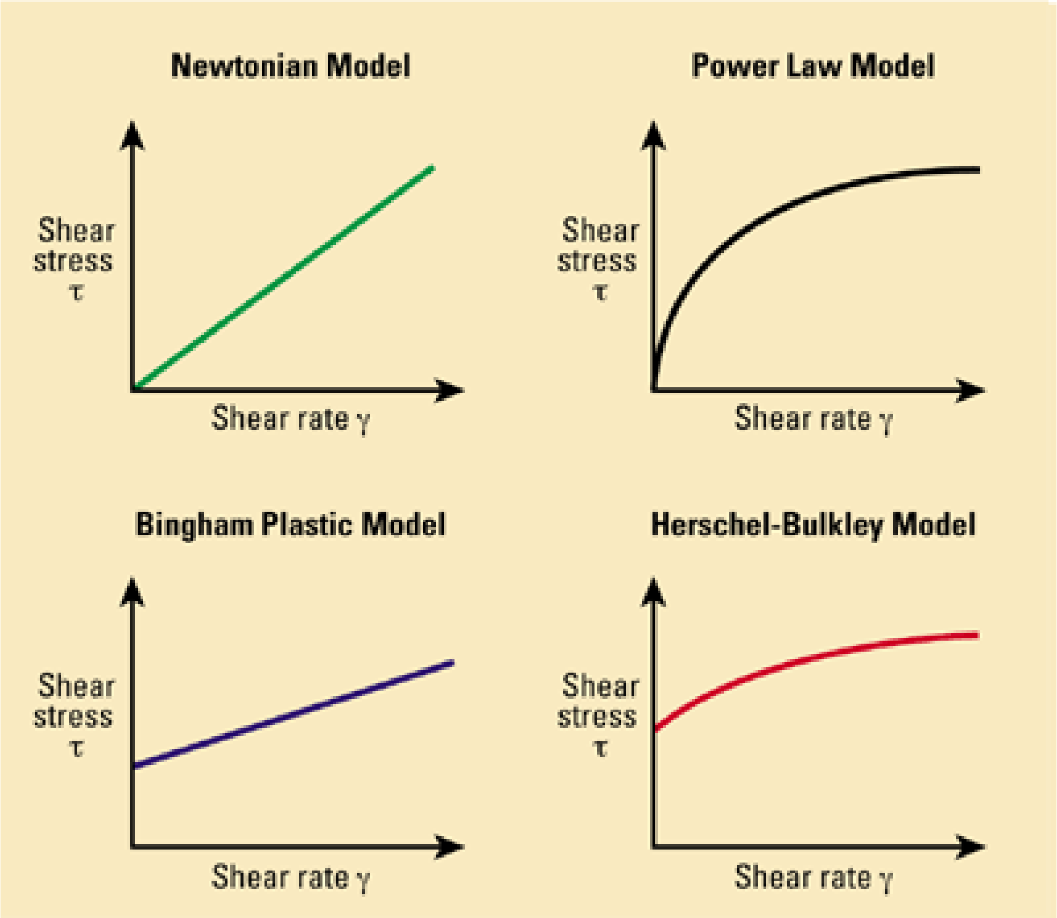
Introduce a **rate-independent initial-yield** in power-law function with $\sigma_e \stackrel{\text{def}}{=} \bar{\sigma} - \sigma_y$:

$$\dot{\bar{\epsilon}}^p = \begin{cases} 0 & \text{if} \quad \sigma_e \leq 0, \\ \dot{\epsilon}_0 \left(\dfrac{\sigma_e}{S} \right)^{(1/m)} & \text{if} \quad \sigma_e > 0. \end{cases}$$

For the case $m = 1$, we have **Bingham Model**:

$$\dot{\bar{\epsilon}}^p = \begin{cases} 0 & \text{if} \quad \sigma_e \leq 0, \\ \dot{\epsilon}_0 \left(\dfrac{\sigma_e}{S} \right) & \text{if} \quad \sigma_e > 0. \end{cases}$$

Rheological Models



1D large-deformation elastic-viscoplastic

$U > 0$,	stretch (l/l_0),
$U = U^e U^p$	elastic-plastic decomposition of U,
U^e	elastic part of the stretch,
U^p,	plastic part of the stretch,
σ,	Cauchy stress.

Strain: $\epsilon^e = \ln U^e$,

Free energy: $\psi^e = \dfrac{1}{2} E \, (\epsilon^e)^2$,

Cauchy stress: $\sigma = E \, \epsilon^e$,

Flowrule : $\dot{U}^p = D^p U^p, \quad D^p = \dot{\bar{\epsilon}}^p \mathrm{sign}(\sigma).$

2D vs. 3D

- General 3D model can be specialized
  - Plane-strain

$$\begin{pmatrix} F_{11} & F_{12} & 0 \\ F_{21} & F_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  - Plane-stress

$$\begin{pmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$