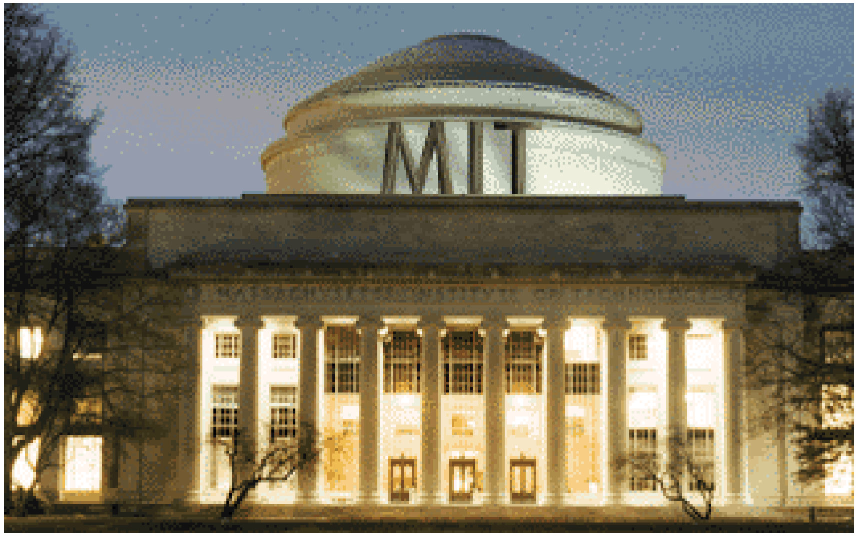
Simple Mechanical Fractional Derivative Models

1) Friedrich, C., H. Schiessel, et al. (1999). "Constitutive behavior modeling and fractional derivatives." Rheology Series: 429-466.

2) Sokolov, I. M., J. Klafter, et al. (2002). "Fractional Kinetics." Physics Today **55(11): 48-54.**

Siddarth Srinivasan

June 18th 2010



Back to RCEs

For $G(t)=\frac{E}{\Gamma(1-\beta)}\left(\frac{t}{\lambda}\right)^{-\beta}$

$$\tau(t)=\frac{E\lambda^{\beta}}{\Gamma(1-\beta)}\int_{-\infty}^{t}dt'(t-t')^{-\beta}\frac{d\gamma(t')}{dt'}$$

CE for assumed form of G

$$\tau(t)=\frac{E\lambda^{\beta}}{\Gamma(-\alpha)}\int_{-\infty}^{t}dt'\frac{1}{(t-t')^{\alpha+1}}\frac{d\gamma(t')}{dt'}$$

Definition of fractional derivative

$$a=-\infty$$
$$\alpha=\beta-1 \quad 0\leq\beta<1$$

$$\tau(t)=E\lambda^{\beta}\frac{d^{\beta-1}}{dt^{\beta-1}}\frac{d\gamma(t)}{dt}$$

$$\tau(t)=E\lambda^{\beta}\frac{d^{\beta}\gamma(t)}{dt^{\beta}}$$

$G^{*}(\omega)=E(i\omega\lambda)^{\beta}$

For SAOS $\gamma(t)=\gamma_0 e^{i\omega t}$

$$\tau(t)=\int_{-\infty}^{t}G(t-t')\frac{d\gamma(t')}{dt'}$$

$$\tilde{\tau}(\omega)=G^{*}(\omega)\tilde{\gamma}(\omega)$$

$$\tilde{\tau}(\omega)=E(i\omega\lambda)^{\beta}\tilde{\gamma}(\omega)$$

Fractional Derivative of a periodic function

$$D^n \sin(x) = \sin\left(x + \frac{n\pi}{2}\right)$$

$$D^\alpha \sin(x) = \sin\left(x + \frac{\alpha\pi}{2}\right)$$

The shift property for the derivative of sine carries over to fractional derivatives

For SAOS

$$\gamma = \gamma_0 \sin(t)$$

$$\tau(t) = E\lambda^\alpha \frac{d^\alpha \gamma(t)}{dt^\alpha} = E\lambda^\alpha \gamma_0 \sin\left(t + \frac{\alpha\pi}{2}\right)$$

Thus, one has a viscoelastic response from fractional RCE in small amplitude oscillatory shear tests

Fractional mechanical element

- represented via hierarchical arrangements: trees, ladders, fractals
- more complex RCE's from combinations of fractional elements

$$\tau(t) = E\lambda^{\beta}\frac{d^{\beta}\gamma(t)}{dt^{\beta}}$$

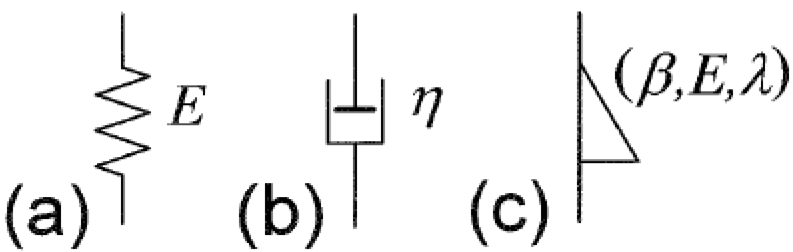


Figure 3. Single elements: (a) elastic, (b) viscous and (c) fractional.

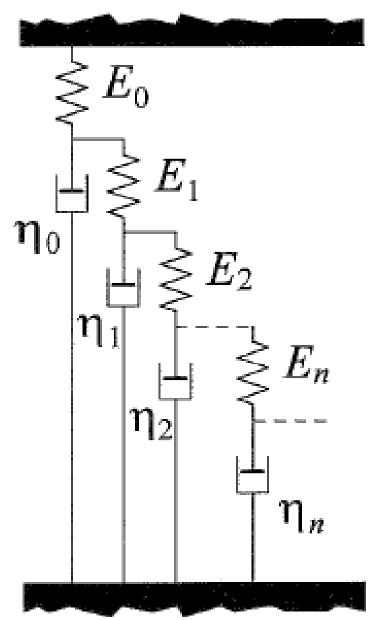
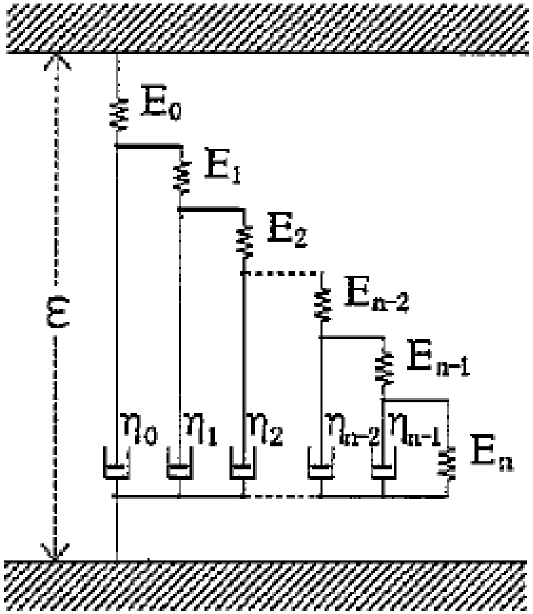


Figure 4. A sequential spring-dashpot realization of the fractional element.

Fractional element is specified by the triplet (β, E, λ)

Images from Friedrich et al. (1999)

Hierarchical analogue to Fractional equations



Mechanical arrangment to simulate FE

$$\varepsilon_k^s, \varepsilon_k^d, \sigma_k^s, \sigma_k^d$$

a) Additivity of strains (series)

$$\varepsilon_k^d = \varepsilon_{k+1}^s + \varepsilon_{k+1}^d \qquad k = 0,1,\ldots,n-2$$

$$\varepsilon = \varepsilon_0^d + \varepsilon_0^s$$

$$\varepsilon_{n-1}^d = \varepsilon_n^s$$

Boundary conditions at upper and lower ends

b) Additivity of stresses (parallel)

$$\sigma_k^s = \sigma_{k+1}^s + \sigma_k^d \qquad k = 0,1,\ldots,n-1$$

$$\sigma = \sigma_0^s$$

Boundary conditions at left end of the ladder

c) Structural elements

$$\varepsilon_k^s = \frac{1}{E_k}\sigma_k^s$$

$$\sigma_k^d = \eta_k \frac{d\varepsilon_k^d}{dt}$$

Schiessel, H. and A. Blumen (1993). "Hierarchical analogues to fractional relaxation equations." Journal of Physics A: Mathematical and General **26: 5057-5069.**

Solution by Fourier transform

$$\varepsilon_k^d = \varepsilon_{k+1}^s + \varepsilon_{k+1}^d \xrightarrow{\varepsilon_k^s = \frac{1}{E_k}\sigma_k^s} \tilde{\varepsilon}_k^d(\omega) = \frac{\tilde{\sigma}_{k+1}^s(\omega)}{E_{k+1}} + \frac{\tilde{\varepsilon}_{k+1}^d(\omega)}{E_{k+1}}$$

rearranging

$$\sigma_k^s = \sigma_{k+1}^s + \sigma_k^d \xrightarrow{\sigma_k^d = \eta_k \frac{d\varepsilon_k^d}{dt}} \tilde{\sigma}_k^s(s) = \tilde{\sigma}_{k+1}^s + (i\omega)\eta_k \tilde{\varepsilon}_k^d(s)$$

$$E_{k+1}\frac{\tilde{\varepsilon}_k^d(\omega)}{\tilde{\sigma}_{k+1}^s(\omega)} = 1 + E_{k+1}\frac{\tilde{\varepsilon}_{k+1}^d(\omega)}{\tilde{\sigma}_{k+1}^s(\omega)}$$

$$E_{k+1}\frac{\tilde{\varepsilon}_k^d(\omega)}{\tilde{\sigma}_{k+1}^s(\omega)} = 1 + \frac{E_{k+1}/\eta_{k+1}}{i\omega + \frac{1}{\eta_{k+1}}\frac{\tilde{\sigma}_{k+2}^s(\omega)}{\tilde{\varepsilon}_{k+1}^d(\omega)}}$$

Recursion relation for k

$$E_{n-1}\frac{\tilde{\varepsilon}_{n-2}^d(\omega)}{\tilde{\sigma}_{n-1}^s(\omega)} = 1 + \frac{E_{n-1}/\eta_{n-1}}{i\omega + E_n/\eta_{n-1}}$$

Special case for k=n-2 (right end)

$$E_0 G^*(\omega) = E_0\frac{\tilde{\varepsilon}(s)}{\tilde{\sigma}(s)} = 1 + \cfrac{E_0/\eta_0}{i\omega + \cfrac{E_1/\eta_0}{1 + \cfrac{E_1/\eta_1}{i\omega + \cfrac{E_2/\eta_1}{1 + \cfrac{E_2/\eta_2}{i\omega + \ldots}}}}}$$

Schiessel, H. and A. Blumen (1993). "Hierarchical analogues to fractional relaxation equations." Journal of Physics A: Mathematical and General **26: 5057-5069.**

Relation between RCE and mechanical model

Exact sum of a binomial series $x(x+1)^{\gamma-1}$ $|x|<1, \gamma \in \mathcal{R}$

$$c_0 = E_0 / \eta_0$$

$$E_1 / \eta_0 = (1-\gamma) c_0$$

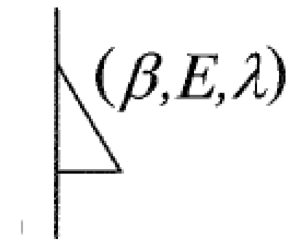
$$E_1 / \eta_1 = \frac{1 \cdot (0+\gamma)}{1.2} c_0$$

simplifies as

$$E_0 G^*(\omega) = 1 + \frac{c_0}{i\omega}\left(\frac{c_0}{i\omega}+1\right)^{\gamma-1} \qquad n \to \infty$$

for small ω ie, $\omega < c_0 \delta^{1/\gamma}$ $\delta \ll 1$

$$E_0 G^*(\omega) = E_0 \frac{\tilde{\varepsilon}(\omega)}{\tilde{\sigma}(\omega)} = (c_0 / i\omega)^{\gamma}$$



$$\tilde{\sigma}(\omega) = E_0 (i\omega\eta_0 / E_0)^{\gamma}\, \tilde{\varepsilon}(\omega) = E_0 (i\omega\lambda)^{\gamma}\, \tilde{\varepsilon}(\omega)$$

IFT ↓

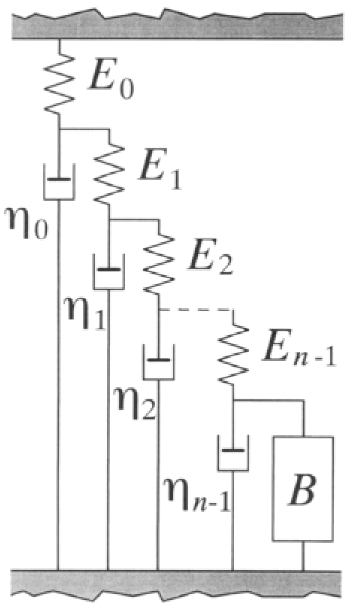
$$\sigma(t) = E_0 \lambda^{\gamma} \frac{d^{\gamma} \varepsilon(t)}{dt^{\gamma}}$$

Within certain limits, hierarchical mechanical model reproduces fractional RCE!

(γ, E_0, λ)

3 parameter FE

Example: Response of gels to shear



- Leaf springs and inelastic blocks

a) pre-gel: finite clusters, terminate ladder with inelastic block

b) critical gel, infinite network

c) post gel: more crosslinks, terminate with leaf springs

Macromolecules, Vol. 28, No. 11, 1995

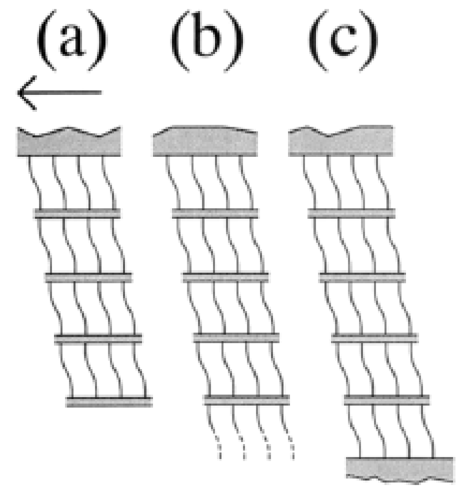
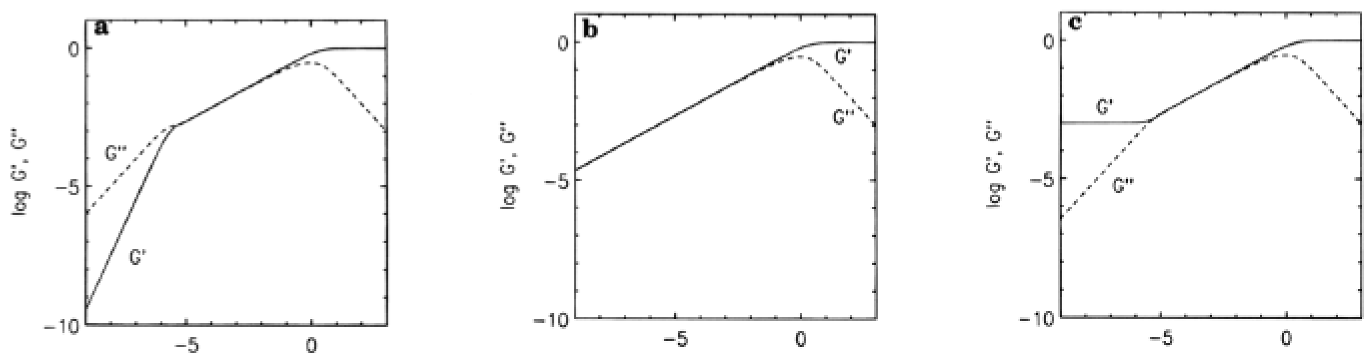


Figure 3. Mechanical analogues of Figure 1, with $n = 3$ for a and $n = 4$ for c. The arrow indicates an external shear load.



η=E=1, n=1000. numerically computed β =1/2 for critical gel. Can modify E, η to tune β <1/2 pre gel and β >1/2 post gel. **β represents structural parameter (connectivity)**

Schiessel, H. and A. Blumen (1995). "Mesoscopic Pictures of the Sol-Gel Transition: Ladder Models and Fractal Networks." <u>Macromolecules</u> **<u>28(11): 4013-4019.</u>**

Fractal Tree Representation of FE

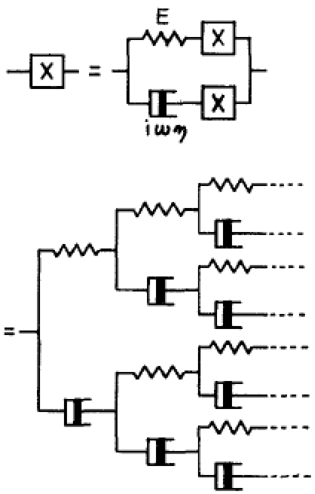


Fig. 1. Self-similar (fractal) tree model for viscoelastic behavior

$$\frac{1}{X} = \left(\frac{1}{E} + \frac{1}{X}\right) + \left(\frac{1}{i\omega\eta} + \frac{1}{X}\right)$$

$$X = E\sqrt{i\omega\lambda}$$

FE with parameters $(1/2, E_0, \lambda)$

For arbitrary fractional order?

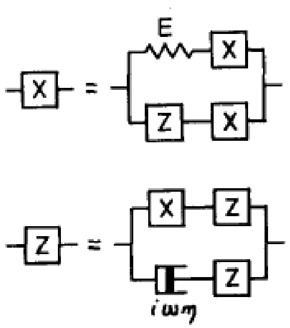


Fig. 2. Two-stage generalization of fractal tree model

$$X = \sqrt{ZE}$$

$$Z = \sqrt{i\omega\eta X}$$

$$X = (i\omega\eta)^{\frac{1}{4}} E^{\frac{1}{2}} X^{\frac{1}{4}}$$

$$X = \sqrt{Z_1 Z_2}$$

iterate

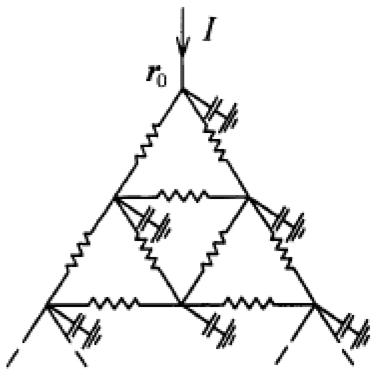
$$X = (i\omega\eta)^{\Sigma(1/2^i)} E^{\Sigma(1/2^j)} X^{1-\Sigma(1/2^i)-\Sigma(1/2\,j)}$$

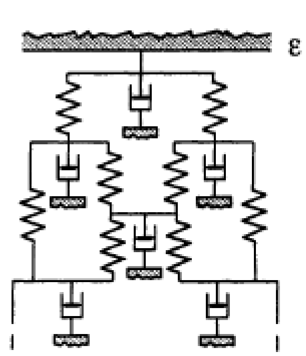
$$X = (i\omega\eta)^{\beta} E^{1-\beta}$$

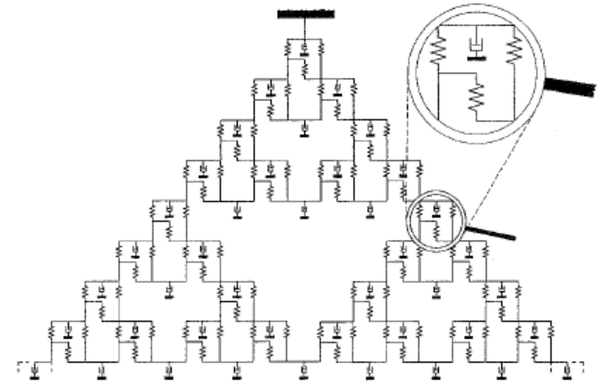
$$\beta = \frac{\Sigma(1/2^i)}{\Sigma(1/2^i) + \Sigma(1/2^j)}$$

Heymans, N. and J. C. Bauwens (1994). "Fractal rheological models and fractional differential equations for viscoelastic behavior." Rheologica Acta **33(3): 210-219.**

Connectivity of mechanical networks







$\beta = 0.317$

nodes in the networks - $\mathbf{r}_j$

each $\mathbf{r}_i$ connected to neighboring $\mathbf{r}_j$ by equal spring (E)

node linked to planar ground with site-dependent viscosity $\eta_i = z(\mathbf{r}_i)\eta$ where $z(\mathbf{r}_i)$ is the coordination number

$$\eta_i \dot{\gamma}_i(t) = E\Sigma_{j(i)}\left[\gamma_j(t) - \gamma_i(t)\right] \xleftrightarrow[\lambda \sim w]{\eta_i \gamma_i(t) \sim \mathrm{P}(\mathrm{r}_i, t)} \frac{dP(r_i,t)}{dt} = \Sigma_{j(i)}\left[w_{ij}P(\mathbf{r}_j,t) - w_{ji}P(\mathbf{r}_i,t)\right]$$

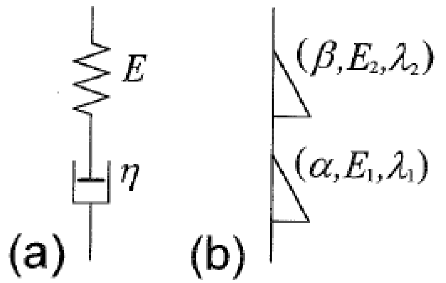
Stress balance on node i *Master diffusion equation*

$$\boxed{\beta = 1 - d_s / 2} \xleftarrow[d_s \text{ is the spectral dimension}]{\text{Tauberian theorems}} P(t) \propto t^{\frac{-d_s}{2}}$$

At large times

Heymans, N. and J. C. Bauwens (1994). "Fractal rheological models and fractional differential equations for viscoelastic behavior." Rheologica Acta **33(3): 210-219.**

Fractional Maxwell & Kelvin-Voigt



Fractional Maxwell Model (FMM)

$$\gamma_1(t) = E_1^{-1}\lambda_1^{-\alpha}\frac{d^{-\alpha}\tau(t)}{dt^{-\alpha}}$$

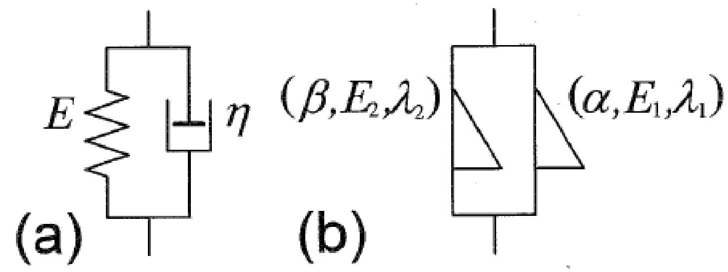
$$\gamma_2(t) = E_2^{-1}\lambda_2^{-\beta}\frac{d^{-\beta}\tau(t)}{dt^{-\beta}}$$

$$\gamma(t) = \gamma_1(t) + \gamma_2(t)$$

$$\tau(t) + \frac{E_1\lambda_1^{\alpha}}{E_2\lambda_2^{\beta}} = E_1\lambda_1^{\alpha}\frac{d^{\alpha}\gamma(t)}{dt^{\alpha}}$$

$$\boxed{\tau(t) + \lambda^{\alpha-\beta}\frac{d^{\alpha-\beta}\tau(t)}{dt^{\alpha-\beta}} = E\lambda^{\alpha}\frac{d^{\alpha}\gamma(t)}{dt^{\alpha}}}$$

Friedrich et al. (1999)



Fractional Kelvin-Voigt Model (FKVM)

$$\tau_1(t) = E_1^{1}\lambda_1^{\alpha}\frac{d^{\alpha}\gamma(t)}{dt^{\alpha}}$$

$$\tau_2(t) = E_2^{1}\lambda_2^{\beta}\frac{d^{\beta}\gamma(t)}{dt^{\beta}}$$

$$\tau(t) = \tau_1(t) + \tau_2(t)$$

$$\tau(t) = E_1^{1}\lambda_1^{\alpha}\frac{d^{\alpha}\gamma(t)}{dt^{\alpha}} + E_2^{1}\lambda_2^{\beta}\frac{d^{\beta}\gamma(t)}{dt^{\beta}}$$

$$\boxed{\tau(t) = E\lambda^{\alpha}\frac{d^{\alpha}\gamma(t)}{dt^{\alpha}} + E\lambda^{\beta}\frac{d^{\beta}\gamma(t)}{dt^{\beta}}}$$

Material Functions of FMM and FKVM

Table 1
Material functions of the fractional Maxwell and Kelvin-Voigt models

	FMM		FKVM	
$G(t)$	$E(t/\lambda)^{-\beta} E_{\alpha-\beta,1-\beta}\left(-(t/\lambda)^{\alpha-\beta}\right)$	(32)	$\frac{E}{\Gamma(1-\alpha)}\left(\frac{t}{\lambda}\right)^{-\alpha} + \frac{E}{\Gamma(1-\beta)}\left(\frac{t}{\lambda}\right)^{-\beta}$	(33)
$J(t)$	$\frac{E^{-1}}{\Gamma(1+\alpha)}\left(\frac{t}{\lambda}\right)^{\alpha} + \frac{E^{-1}}{\Gamma(1+\beta)}\left(\frac{t}{\lambda}\right)^{\beta}$	(34)	$E^{-1}(t/\lambda)^{\alpha} E_{\alpha-\beta,1+\alpha}\left(-(t/\lambda)^{\alpha-\beta}\right)$	(35)
$G^*(\omega)$	$E\frac{(i\omega\lambda)^{\alpha}}{1+(i\omega\lambda)^{\alpha-\beta}}$	(36)	$E(i\omega\lambda)^{\alpha} + E(i\omega\lambda)^{\beta}$	(37)
$J^*(\omega)$	$E^{-1}(i\omega\lambda)^{-\alpha} + E^{-1}(i\omega\lambda)^{-\beta}$	(38)	$E^{-1}\frac{(i\omega\lambda)^{-\beta}}{1+(i\omega\lambda)^{\alpha-\beta}}$	(39)

Table from Friedrich et al. (1999)

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

Generalized Mittag-Lefler function of argument (z)

-FMM and FKVM : two power law regions intersecting at t=λ

- For G(t) of FMM has flat decrease of slope (-β) followed by faster decrease (-α)

- For G(t) of FKVM, fast decrease followed by flat decrease

FMM and FKVM

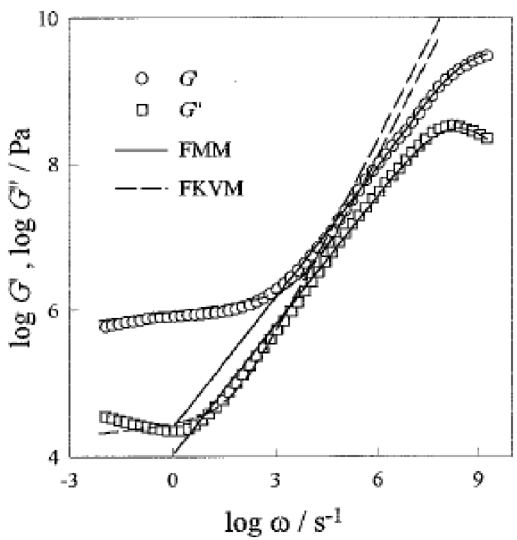


Figure 10. Description of the data of Figure 1(a) by the fractional Maxwell model (solid lines) and the fractional Kelvin-Voigt model (dashed lines).

Table 2
Material parameters used in Figure 10

	log E	log λ	α	β
FMM	9.09	-8.36	0.583	0.593
FKVM	5.75	-2.96	0.057	0.885

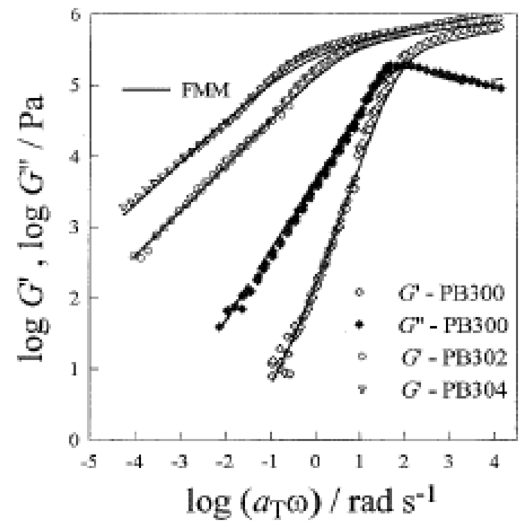


Figure 11. Description of the data of Figure 2 by the fractional Maxwell model (solid lines).

Table 3
Material parameters used in Figure 11

	log E	log λ	α	β
PB300	5.52	-1.87	0.882	0.994
PB302	5.60	-0.344	0.553	0.632
PB304	5.48	0.720	0.478	0.590

- polyisobutylene: FKVM unable to fit G' and G'' in glassy and transitional zones

- modified polybutadiene's terminal relaxation zone described by FMM

- 4 fitting parameters?

Images from Friedrich et al. (1999)

3 element models

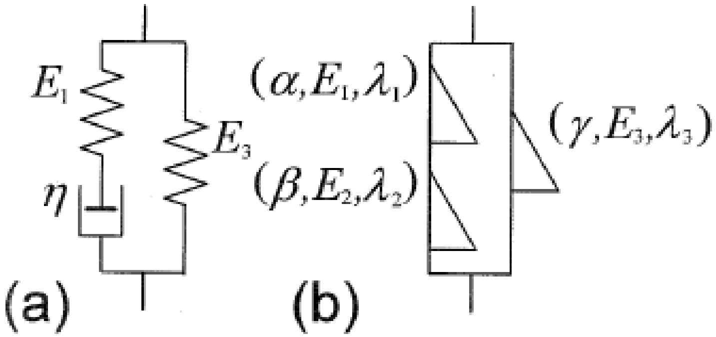


Figure 13. The (a) ordinary and (b) fractional Zener model.

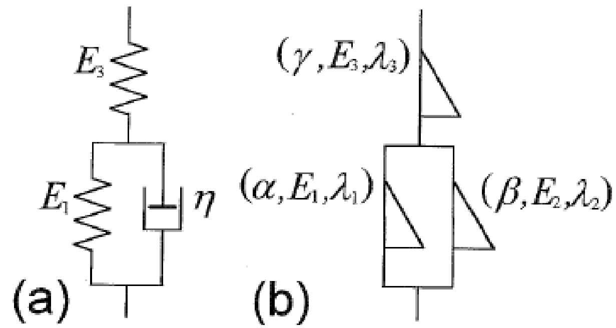


Figure 14. The (a) Poynting-Thomson model and (b) its fractional generalization.

The FMM and FKVM cannot describe ‘S’ shaped transitions in G(t)

3-Fractional Element models: the *Fractional Zerner Model* (FZM) and the *Fractional Poynting-Thompson Model* (FPTM)

FCM RCE

$$\tau(t) + \lambda^{\alpha-\beta} \frac{d^{\alpha-\beta}\tau(t)}{dt^{\alpha}} = E\lambda^{\alpha} \frac{d^{\alpha}\gamma(t)}{dt^{\alpha}} + E_0\lambda^{\gamma} \frac{d^{\gamma}\gamma(t)}{dt^{\gamma}} + E_0\lambda^{\gamma+\alpha-\beta} \frac{d^{\gamma+\alpha-\beta}\gamma(t)}{dt^{\gamma+\alpha-\beta}}$$

$$G(t) \propto \begin{cases} t^{-\beta} & \text{for } t \ll \lambda \\ t^{-\alpha} & \text{for } \lambda \ll t \ll \lambda_1 \\ t^{-\gamma} & \text{for } \lambda_1 \ll t\ . \end{cases}$$

FPTM RCE

$$\tau(t) + \frac{E}{E_0}\lambda^{\alpha-\gamma} \frac{d^{\alpha-\gamma}\tau(t)}{dt^{\alpha-\gamma}} + \frac{E}{E_0}\lambda^{\beta-\gamma} \frac{d^{\beta-\gamma}\tau(t)}{dt^{\beta-\gamma}} = E\lambda^{\alpha} \frac{d^{\alpha}\gamma(t)}{dt^{\alpha}} + E\lambda^{\beta} \frac{d^{\beta}\gamma(t)}{dt^{\beta}}$$

λ_1 , E_0 are parameter groupings